H. Ruddat, Log GW invariants II 10/28/2016

f homogeneous poly. of degree 5.

Consider $X_t := \{z_0 \cdots z_4 + tf = 0\} \subseteq \mathbb{P}^4$. There are 2875 lines in the quintic; how to see this via degenerating to X_0?

[Katz–Nishinou]'s: First, prove

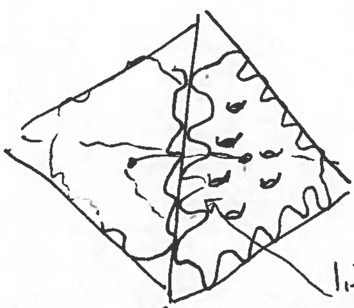
Lemma: The line in X_0 deforms into $X_{t\neq 0}$ $\iff$ it meets $\operatorname{Sing} X_0 = \bigcup^{10} \mathbb{P}^2$ discretely (only) in $\operatorname{Sing} X_0 \cap \{f=0\}$, "some kind of union of quintic curves", but not any coord. $\mathbb{P}^1$.

(*) (& not contained in any $\mathbb{P}^2$!) implied by

A picture:

component of X_0 (5 of them):

$\mathbb{P}^3 \subseteq \mathbb{P}^4$.



$(f=0) \cap \mathbb{P}^3$ is a quintic in $\mathbb{P}^3$.

want $(f=0) \cap \partial\mathbb{P}^3$ ← toric boundary.

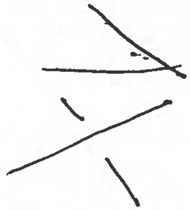
line needs to meet each component of $(f=0) \cap \partial \mathbb{P}^3$.

coord. $\mathbb{P}^1$ - (can't meet these!)

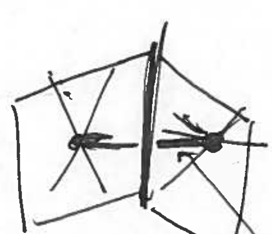
Figure out how many of (*) there are.

warmup

Q: How many lines meet 4 given lines in $\mathbb{P}^3$? $\boxed{2}$



$\longrightarrow$ deform to two planes



the other line

one line (intersection of planes)

Philosophy [Schubert]: As one deforms lines, the count either stays same, or becomes infinite.

Q: How many lines in $\mathbb{P}^3$ meet 4 plane quintics?

$\underline{2} \cdot 5 \cdot 5 \cdot 5 \cdot 5$. ← ans. from before

Need to compensate!

(a) don't want lines to intersect $\mathbb{P}^1$'s

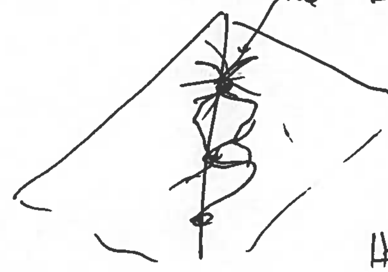
(b) strictly speaking, not in general position (but by Schubert philosophy, should be ok)

$\#\text{lines} = 5 \cdot \big((2 \cdot 5^4) -$ those meeting the $\mathbb{P}^1$'s $\big)$.

↑ $\#\mathbb{P}^3$ components

↑ how many of these are there?

need to meet one of these points; the space of such lines is a $\mathbb{P}^2$. "sphere of vision"



How many meet the other two quintics? (project two quintics into $\mathbb{P}^2$, Bezout $\Rightarrow$ there are $\boxed{25}$ points where two quintics intersect "sphere of vision".)

Ans. to is:

25 ~~quintics~~ lines / pt. of intersection of two quintics

$\times$ 30 # pts. p in $\{f=0\} \cap \mathbb{P}^1$

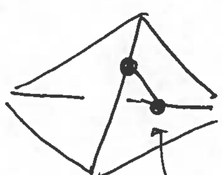
(5 intersections per $\mathbb{P}^1$ * 6 $\mathbb{P}^1$'s).

$-$ double counts

$= 3 \cdot 5 \cdot 5 = 3 \cdot 25$.

↑ # pairs of opposite $\mathbb{P}^1$'s ← (unless opposite, line will be contained in a $\mathbb{P}^2$! and/or won't intersect all 4 quintics.)

↑ quintic ↑ quintic.



line between two $\mathbb{P}^1$'s get counted twice!

Hence, $\#\text{lines} = 5 \cdot (2 \cdot 5^4 - (30-3)25) = 2875$

Also want higher degree curves e.g. conics.

Relies on a general degeneration formula; ongoing work of [Abramovich-Chen-Gross-Siebert]

[Kim-Lho

~~&~~ Chen '14

$X_0 \subseteq \mathbb{P}^4$ is normal crossing, but

$X_0 \subseteq \mathcal{X} = \{ \underline{tf + z_0 \cdots z_4} = 0 \} \subseteq \mathbb{P}^3 \times \mathbb{A}^1_t$ is not sensible because $\mathcal{X}$ is singular.

locally $tf + \underset{\sim}{xy} = 0 \sim z_1^2 + \cdots + z_4^2 = 0$ $(t = xyzw$ would have been ok$)$.

quintic in $\mathbb{P}^2$ not invertible. $=0$ on $\mathbb{P}^2$ node locus.

Blow up successively components of X_0 in $\mathfrak{X}$. "small resolution;" (add only stuff of codim. 2)
does nothing to
Cartier divisor points,
but deals w/ singular
points well!)

(Rmk: Given

$$X \supseteq Z.$$

$Bl_Z X$ satisfies univ. property that

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[dr, dashed] & X \supseteq Z \\ & Bl_Z X \arrow[u] \end{tikzcd}$$

whenever $\pi^{-1}(Z)$ is a Cartier divisor.

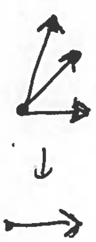
(so if Z is Cartier, $Bl_Z X$ does nothing))!

The situation of the blow up is a little asymmetric (have to choose order of points to blow-up, answer depends on this).

$\rightsquigarrow$ $\tilde{\mathfrak{X}}$ normal crossings degeneration which is semistable i.e. locally $t = xyzw$

$\downarrow$ $\Rightarrow$ log smooth; locally in toric charts, looks like:

$\mathbb{A}^1$



$$\begin{tikzcd} \mathbb{N}^n \arrow[r] & k[x_1, \ldots, x_n] & x_1 \cdots x_n \\ \mathbb{N} \arrow[r] \arrow[u] & k[t] \arrow[u] & t \arrow[u, mapsto] \end{tikzcd}$$

$e_i \mapsto x_i$, $1 \mapsto t$.

or equivalently $\Longleftrightarrow \mathcal{M}(\tilde{\mathfrak{X}}, \tilde{X}_0)$. (c.f. last talk, $\mathcal{M}(X, D)$).

Def: Fix $X \to S$ log smooth map. A log stable map is a commutative diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\text{log smooth}"'] & X \arrow[d] \\ W \arrow[r] & S \end{tikzcd}$$

of log schemes, such that without log structures, have an ordinary stable map.

($\Rightarrow$ $C \downarrow W$ dim. 1).

W: param. scheme; S: could be family or pt.

Note: log smoothness of $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ $\Rightarrow$ $\Omega_{X^{\log}/S^{\log}}$ is locally free

& $H^0(C, f^*(\Omega_{X^{\log}/S^{\log}})^\vee)$ classifies 1st order deformations of log maps ← tangent sheaf.

$\Rightarrow$ perfect obstruction theory $\Rightarrow$ virtual fundamental class.

(Rmk: the underlying $\Omega_{X/S}$ (non-log version) may be highly non-locally free!)

$\rightsquigarrow$ $\widetilde{\mathcal{M}}(\widetilde{X}_0/0)$ = moduli space of log stable maps.

(it's a category/stack;)

morphisms are

$$\begin{tikzcd}
C \arrow[rr, "f"] \arrow[dr] \arrow[dd, "\log \text{ smooth}"'] & & X \arrow[dd] \\
& C' \arrow[ur] \arrow[dd] & \\
W \arrow[rr] \arrow[dr] & & S \\
& W' \arrow[ur] &
\end{tikzcd}$$

problem: it's not separated! But, have

"loc. finite type")

$\widetilde{\mathcal{M}}(\widetilde{X}_0/0)$

$\cup|$ open subset

$\mathcal{M}(X_0/0)$ = basic / minimal log stable maps.

[Gross-Siebert] ↕ basic

minimal ↕ [Abramovich-Chen]

If log space is "combinatorially fin. type," $\mathcal{M}(X_0/0)$ will will be finite type.

Degeneration formula: basic idea:

may have had to blow up.

want: $$[\mathcal{M}(\tilde{X}_{0/0})]^{vir} \underset{\text{split}}{=} \prod_{\mathbb{P}^3} [\mathcal{M}(\tilde{\mathbb{P}}^3, \partial\tilde{\mathbb{P}}^3)]^{vir}$$

$(**)$

+ need Base change formula:

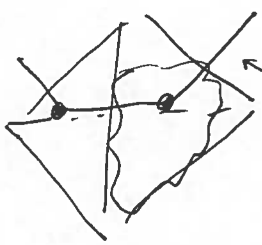
Thm: "Log GW invariants are constant in log smooth families."

[Mandel-R.]: (uses in a large way Behrend–Fantechi)

Together, this would imply that curve counts in $\tilde{X}_t$ come from these.

"split curves"

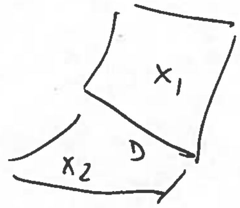
Rmk: A log stable map if it goes into a "a.c. singular point", needs to "keep going" must go on. (In a sense, this explains why one can split nicely).



split in 3-pieces.

stays in one component. etc.

Q: where does $(**)$ live?



$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{M}(X_1)\times\mathcal{M}(X_2) \\ \downarrow & & \downarrow \text{ev.} \\ D & \xrightarrow{\text{diag}} & D^2 \end{array}$$

want $[\mathcal{M}]^{vir} = \text{diag}^*\left([\mathcal{M}(X_1)]\times[\mathcal{M}(X_2)]\right)$.