J. Evans — Lag'n cell cplxes & Markov numbers — PU/IAS symplectic seminar — 9/20/2016

joint w/ I. Smith

$X \subseteq \mathbb{CP}^N$

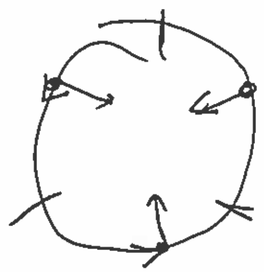
If X_t is a family of proj. varieties s.t. X_0 singular, some subset of X_1 gets crushed onto a sing'y!

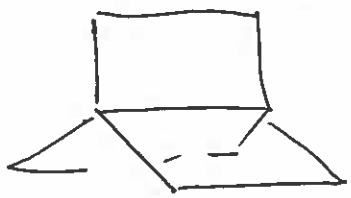
∥ transport to singularity

$\not H \quad X.$

A "p-pinwheel" is a 2-d cell cplex

$p=2$





Def: A Lag'n pinwheel is an immersion

$$i : D^2 \longrightarrow (X, \omega)$$

s.t. $i^*\omega = 0$

& s.t. $i|_{\mathring{D}^2}$ is embedding

& $i(D)$ is homeo. to pinwheel.

This is the vanishing cycle of a Wahl singularity, surface singularity:

$\mathbb{C}^2 / \Gamma_{p,q}$ acting by $(x,y) \longmapsto (\eta x, \eta^q y)$, $\gcd(p,q) = 1$.

$\Gamma_{p,q}$: action of $\{\eta \mid \eta^{p^2} = 1\}$

smoothing is a Lag'n pinwheel.

(there's a unique smoothing component; in general very good geometry!)

The Milnor fibre of this, $B_{p,q}$, is a Stein domain whose Lag'n skeleton $L_{p,q}$ is a p-pinwheel.

$\partial B_{p,q} = \Sigma_{p,q} = L(p^2, pq-1)$ Lens space; and

$$H_*(B_{p,q}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0 \\ 0 & \text{else.} \end{cases}$$

rational ball.

(come up a lot in 4-manifold topology, in alg. geometry as moduli of gen. type surfaces w/ "semi-log-canonical singularity" compactification.

Rmk: ~~[illegible]~~ $B_{p,q}$, as q vary, realize many local Lag'n models for nbhd of a p-pinwheel.

Thm: (Khodorovskiy): Any Lag'n p-pinwheel is contained in a symplectically embedded $B_{p,q}$ (of some volume) for some q, $1 \le q < p$; & $B_{p,q} \cong B_{p,p-q}$.

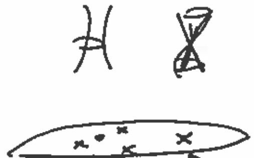
How to determine q?

Some winding # near ○ of singularities telling you how the tangent planes wind along the singular circle, & only residue mod p matters.

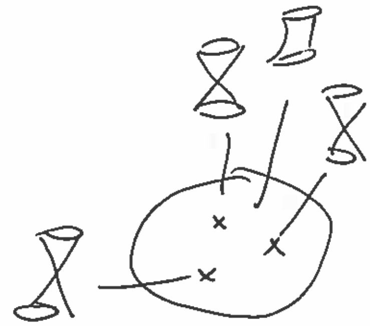
Another description of $B_{p,q}$:

$$A_{p-1} = \{xy - z^p = 1\} \subseteq \mathbb{C}^3 \quad \text{smooth surface.}$$

$$\begin{array}{c} A_{p-1} \\ \big\downarrow z \\ \mathbb{C} \end{array}$$



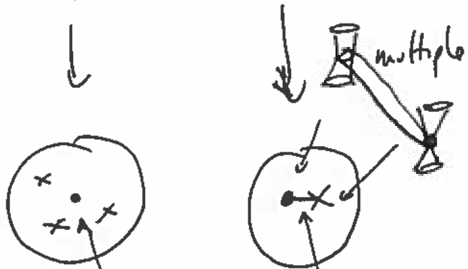
wherever $z^p = 1$



$$A_{p-1}/(\mathbb{Z}/p) = B_{p,q}$$

where the action is $(x,y,z) \longmapsto (\eta x, \eta^{-1} y, \eta^{q} z)$, $\eta^p = 1$. (note this preserves eq'n for A_{p-1}).

$$\begin{array}{ccc} A_{p-1} & \longrightarrow & B_{p,q} \\ \big\downarrow & & \big\downarrow \end{array}$$



multiple fiber above 0.

0 fixed by action.

this arc is the projection of a Lagrangian pinwheel $L_{p,q}$.

[Lekili–Maydanskiy] $B_{p,q}$ contains no exact Lag'n submanifolds.

Thm: [E-Smith]: Suppose L_{p_i,q_i}, $i = 1, \dots, N$ pairwise disjoint Lag'n pinwheels in $\mathbb{CP}^2$. Then $\{p_1, \dots, p_N\}$ is a subset of a Markov triple. (In particular, $N \le 3$).

Moreover, $q_i = \pm 3\dfrac{p_j}{p_k}$ mod p_i where i, j, k are different.

A Markov triple is a triple $a,b,c \in \mathbb{N}$ s.t.

$$a^2+b^2+c^2 = 3abc.$$

e.g., $(1,1,1)$.

Rewrite this eq'n as $a^2+b^2 = (3ab-c)c$.

- any solution of the form $(a,b,c) \rightsquigarrow$ a solution $(a,b,3ab-c)$ <u>mutation</u>

induces, by picking one of the variables to be "c".

$(1,1,1) \longrightarrow (1,1,2)$

$\downarrow$ mutate on 1

$(1,2,5) \longrightarrow (2,5,29) \nearrow \searrow$

$\searrow (1,5,13) \nearrow \searrow$

A Markov <u>number</u> is any element of a Markov triple.

Claim: all solutions are linked to $(1,1,1)$ by <u>mutation</u>.

Note: $a^2+b^2 = (3ab-c)c$, so if c is a Markov number

$\Rightarrow$ any odd factor of $c \equiv 1 \mod 4$.

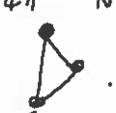
(b/c it's a sum of squares!)

So for instance $L_{3,q}, L_{7,q}, L_{11,q} \ldots \subseteq \mathbb{CP}^2$.

(Rmk: Markov numbers are <u>very sparse</u>)

Note: $L_{13,2} \subseteq \mathbb{CP}^2$, $L_{29,7} \subseteq \mathbb{CP}^2$ but $L_{13,2} \cap L_{29,7} \neq \emptyset$, up to Hamiltonian deformation.

<u>Thm</u>: (Hacking–Prokhorov): For any Markov triple (a,b,c), there is a degeneration of $\mathbb{CP}^2$ to a toric algebraic surface $\mathbb{CP}^2(a^2,b^2,c^2)$ with Wahl singularities modeled on $\mathbb{C}^2/\Gamma_{a,2a}$, etc.



(<u>Rmk</u>: there's also work on exceptional collections of vector bundles on $\mathbb{CP}^2$ & Markov triples .. what's the relation?)

(Moreover, any <u>$\mathbb{Q}$-gorenstein</u> degeneration of $\mathbb{CP}^2$ (restriction on whole family)

$\mathcal{X}$
$\downarrow$
$\mathbb{C}$

has $\mathcal{X}_0 = \mathbb{CP}^2(a^2,b^2,c^2)$ for some $a^2+b^2+c^2=3abc$.

So, you <u>can</u> find pinwheels for every Markov triple.

(& algebraically, that's it. Turns out, symplectically, that's it too)

Rmk: Vianna's tori also comes from these degenerations.

(ex: (1,1,2): central fibre is T^2_{Chek}, pinwheel is $\mathbb{RP}^2$,

$\mathcal{B}$ mod 2, T^2_{Chek} missing a critical fibre; $\mathbb{RP}^2$ generates missing critical value.

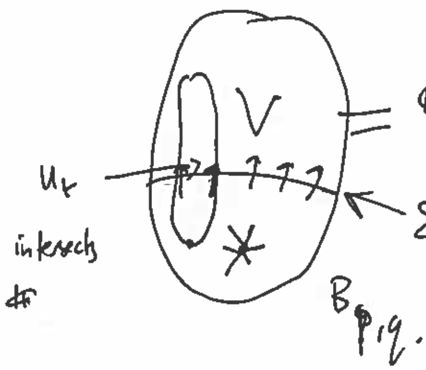
Expect same in other instances for Vianna's tori $\mathcal{B}$ these $L_{p,q}$'s (if they could be made sense of in the Fukaya category.

["cf. "Unicity conj. for Markov triples" : largest # determines other two."]

unusual that one can come up w/ a Markov number w/ other triple.

Proof: $N=1$ case.

if $L_{p,q} \subseteq \mathbb{CP}^2$ then $\exists a, b$ s.t. $p^2+a^2+b^2=3apb$



$= \mathbb{CP}^2 \setminus B_{p,q}$

u_t a J_t-hol. line.

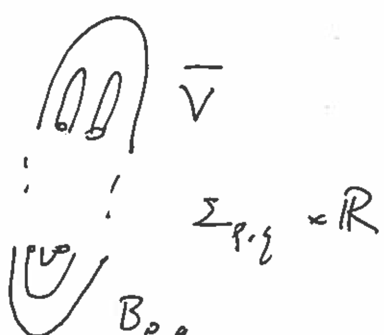
s.t. $[L_{p,q}] \in H_2(\mathbb{CP}^2; \mathbb{Z}/p)$ (this homo. class actually makes sense)

$L_{p,q} \cap u_t \neq 0$.

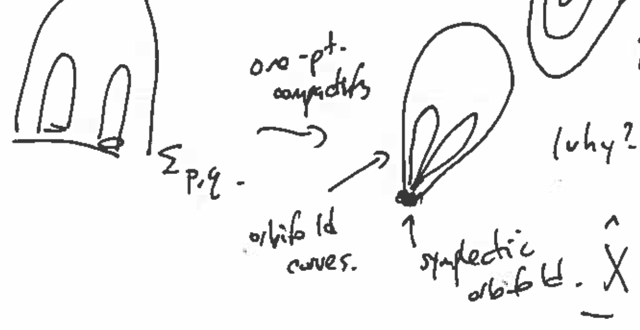
J_t ~~parameters~~ a "neck-stretching family".

In the limit:

Look at $\bar{V}$:



SFT compactness [BEHWZ]



(why? Recall $\Sigma_{p,q} = S^3/\Gamma_{p,q} \subseteq \mathbb{C}^2/\Gamma_{p,q}$.
use the integrable almost complex structure, so a nbhd of $\Sigma_{p,q}$ looks like a nbhd of 0. orbifold cpctify).

Prop: One of the components of this SFT limit is an orbifold sphere with one orbifold point (isotropy $\mathbb{Z}/p^2$) in moduli space of $\mathrm{vdim}_{\mathbb{C}} = 1$.

(not surprising b/c space of lines in $\mathbb{CP}^2$ is high-dimensional).

In a neighborhood of orbifold point (lifting to $\mathbb{C}^2$), using integrable str.:

$$z \longmapsto \left(z^{Q}, \sum_{i=1}^{\infty} a_i z^{R_i}\right) \qquad \gcd(Q, R_i) = 1.$$

$$Q = q^2 \qquad R_1 = b^2$$

So, asymptotic Reeb orbit is

Cone (a^2, b^2) torus knot

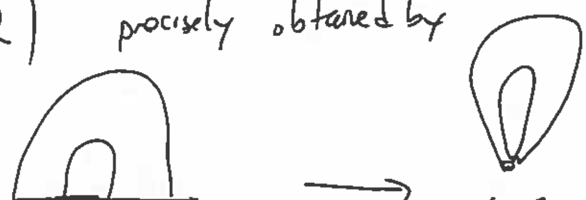
in $S^3/\Gamma_{p,q}$.

Rmk: [Hofer]: Q, R_1 can be entirely phrased in terms of the SFT / Siefring theory of $\Sigma_{p,q}$. (winding + asymptotic operator, ...)

$\exists$ class $\mathcal{E} \in H_2(\hat{X}; \mathbb{Q})$ orbifold

s.t. $\mathcal{E}^2 = \frac{1}{p^2}$, generating

a copy of $\mathbb{Z}$ in $H_2(\hat{X}; \mathbb{Q})$ precisely obtained by



$\mathbb{Z} \subseteq \mathbb{Q}$

$\mathbb{Z}$ not hom. class

all our curves $C = D\mathcal{E}$ $\quad D \in \mathbb{Z}$.

In particular, $c_1(\hat{X}) = 3p\mathcal{E}$ (think $p\mathcal{E} = H$ hyperplane class in $\mathbb{CP}^2$)

integer class,

Adjunction formula tells us:

$$c_1(\hat{X}) \cdot C = C \cdot C + \chi(C) + \text{corrections.}$$

$$\frac{3pD}{p^2} = \frac{D^2}{p^2} + 2 - \left(1 - \frac{1}{p^2}\right) + \frac{(Q-1)(R_1-1)}{p^2}$$

sphere so $\chi = 2$ $\quad$ correction b/c of $\frac{1}{?}$ orbifold point

$$\Rightarrow \cancel{D^2} - 3_p D + p^2 - \overbrace{\cancel{QR}}^{\text{cancel}} + Q + R = 0.$$

And now prove: $\underline{D^2} = \underset{a^2}{Q}\,\underset{b^2}{R}$, by using intersection w/ a pushoff (b/c $\mathrm{vol}_{\mathbb{C}} = 1$)

Q, R coprime $\Rightarrow$ both have to be squares

$$\rightsquigarrow \quad p^2 + a^2 + b^2 = 3abp.$$

(OR: can also use Siefring's adjunction formula.).

promising idea: use Siefring theory to deal w/ many other interesting cases.

Same sort of tricks work for $\mathbb{P}^1 \times \mathbb{P}^1$; though don't get a Markov equation exactly (that's special to $\mathbb{P}^2$).