K. Fukaya, Constructing VFCs. 10/4/2016

$\left\{ \begin{array}{c} \mathcal{M} \\ \text{moduli space (of} \\ \text{J-hol. curves)} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{c} \mathcal{M} \text{ has} \\ \text{"Kuranishi} \\ \text{structure"} \end{array} \right\} \longrightarrow$ ~~Define~~ $[\mathcal{M}]$ fund. class / chain of $\mathcal{M}$.

(prove cpct), relatively less well packaged

in current formulation, this is not completely canonical, depends on choice of extra data here,

this is rel. well packaged

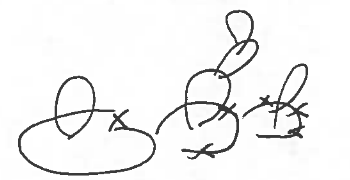
Fix a specific choice: (X^{2n}, ω, J), X convex (or cpct), J compatible. $L^n \subset X^{2n}$ compact Lag'n, $\beta \in H_2(X, L; \mathbb{Z})$

$\vec{z} = (z_1, \ldots, z_k) \in \partial\Sigma$ cyc. ordered.

$\mathcal{M}_{k+1}(\beta)$ $k \in \mathbb{Z}_{\geq 0}$: space of $(\Sigma, \vec{z}_i, u)$ $u : (\Sigma, \partial\Sigma) \to (X, L)$ holo.

Σ bordered stable curve

genus 0, 1 bdry component



(Stage 1: know $\mathcal{M}_{k+1}(\beta)$ cpct).

A Kuranishi chart: given $p \in \mathcal{M}_{k+1}(\beta)$,

(U_p, E_p, s_p, ψ)

U orbifold, $E \to U$ vec. bdle,

s : section, & $\psi : s^{-1}(0) \to \mathcal{M}_{k+1}(\beta)$ homeo. to a nbhd. of p.

U_p is basically $\{ (\Sigma', \vec{z}', u') \text{ s.t. } \bar{\partial} u' \equiv E_p(u'),$

where $E_p(u') \subseteq C^\infty(u'^* TX \otimes \Lambda^{0,1}) \}$.

Need: for each p &

$x = (\Sigma', \vec{z}', u')$ "close to p" (except not necessarily J-hol.;

Fix $E_p(x) \subseteq C^\infty(u'^* TX \otimes_{\mathbb{C}} \Lambda^{0,1})$.

Obstruction data:

$\forall\, p \in \mathcal{M}_{k+1}(\beta)$, & for any $x = (\Sigma', \vec{z}', u')$ close to p,

need to give $E_p(x) \subseteq C^\infty(\Sigma', u'^* TX \otimes \Lambda^{0,1})$ finite dim'l subspace satisfying:

① semi-continuous ② exponential decay ③ transversal, then

(need Thm: $\exists$ at least one $E_{\mathbb{P}}$).

$\Rightarrow$ $\mathcal{M}_{\text{unv}}(\beta)$ has a Kuranishi structure & its germ depends only on $E_{\mathbb{P}}(x)$.

$\uparrow$ $\boxed{\text{Analysis}}$

(Rmk: germ of a Kur. chart dangerous, but germ of a K. str. is ok)

Prop: If have two choices w/

$$E^1_{\mathbb{P}}(x) \subset E^2_{\mathbb{P}}(x)$$

$\Rightarrow$ $\mathcal{U}^1, \mathcal{U}^2$ of $\mathcal{M}_{\text{unv}}(\beta)$ K str. & $\mathcal{U}^1 \hookrightarrow \mathcal{U}^2$ emb. of K. str.

Cor: $\mathcal{U}^1, \mathcal{U}^2$ are cobordant.

Def: obstruction data $E_p \sim E_p'$ $\overset{\text{means}}{\Longleftrightarrow}$ $E_p \subset E_p^1 \supset E_p^2 \subset \cdots$ $\cdots \subset E_p^k \supset E_p'$. (zig zag)

Prop: Any two E_p, E_p' are equivalent.

The proposition states, any two are equivalent.

(expect: & $\exists$ a contractible choice of equivalences, & can make a 'universal choice' of all possible data to have sth. independent of choice).

Cor: any two $\mathcal{U}$'s are cobordant.

Application: Say $X \circlearrowleft G$ G compact, preserving ω, J & suppose $L \circlearrowleft G$.

Def: $E_p(x)$ is G-equivariant.

$$\Leftrightarrow g^* E_{gp}(gx) = E_{\mathbb{P}}(x).$$

where, if $\mathbb{P} = (\Sigma, \vec{z}, u)$

$$g\mathbb{P} = (\Sigma, \vec{z}, g\cdot u)$$

& $g^* : C^\infty(\Sigma', (gu')^* TX \otimes \Lambda^{0,1}) \to C^\infty(\Sigma', (u')^* TX \otimes \Lambda^{0,1})$.

Thm: If: $E_{\mathbb{P}}(x)$ is G-equivariant, then the corresp. Kur. str. is G-equiv.

$(U_{\mathbb{P}}, E_{\mathbb{P}}, s_{\mathbb{P}}, \psi_{\mathbb{P}})$ note: $\mathrm{Im}\,\psi_{\mathbb{P}} \supset G_{\mathbb{P}}$, so these charts are a bit bigger.

Non-trivial: to find such a system of $E_{\mathbb{P}}(x)$; complicated e.g., for toric manifolds.

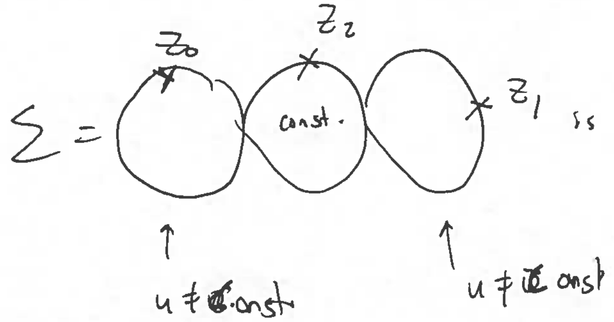
Another type of symmetry

forget:

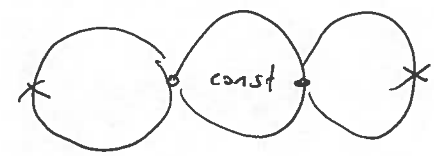
$\mathcal{M}_{k+1}(\beta) \longrightarrow \mathcal{M}_k(\beta)$ stabilize.

$(\Sigma, (z_0, \dots, z_k), u) \longmapsto (\bar{\Sigma}, (z_0, \dots, z_{k-1}), u)$

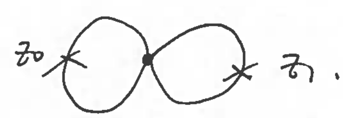
ex:

$\Sigma =$ 

is stable. forgetting z_2 gives



stabilize



$\bar{\Sigma}$.

$E_{\mathbb{P}}$ is obst. ~~data~~ data on $\mathcal{M}_k(\beta)$

$\Rightarrow$ we can pull it back to $\mathcal{M}_{k+1}(\beta)$

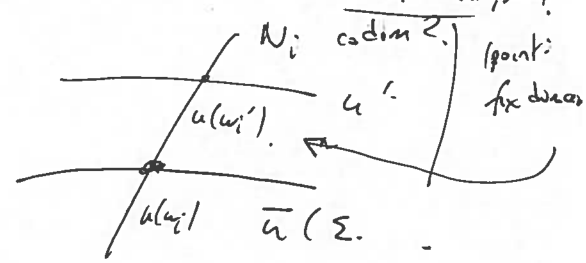
$\mathbb{P} \in \mathcal{M}_{k+1}(\beta) \rightsquigarrow \mathrm{forget}(\mathbb{P}) = \mathbb{q} \in \mathcal{M}_k(\beta)$.

$\mathbb{X}$ is close to $\mathbb{P}$.

$\mathbb{P}$ + "stabilization data" (making source curve stable.) allows one to talk about a map $\mathbb{X}$ being "close to $(\mathbb{P}, st.)$"

$(\Sigma, \vec{z}, u)$ stab. data := $\vec{w}$ s.t. $(\Sigma, \vec{z} + \vec{w}, u)$ s.t. ...

$\vec{w} \ni w_i$



Lemma: for each st_1, st_2, ε $\exists \delta$ s.t.

if $\mathbb{X}$ is δ close to $(\mathbb{P}, st_1)$

$\Rightarrow$ $\mathbb{X}$ is ε-close to $(\mathbb{P}, st_2)$.

(details only talk about $\mathbb{X}$ being close to $\mathbb{p}$, not $\mathbb{p}$ being close to $\mathbb{X}$ ($\mathbb{X}$ is not J-hol.))

Also: If $\mathbb{X}$ is δ-close to $\mathbb{P}$, then

δ close gives obstruction bundles

forget $(\mathbb{X})$ is ε-close to forget $(\mathbb{P})$.

$\Rightarrow$ can pull back obstruction bundles. (what say if $\mathbb{P}$ has const. components

$\mathbb{X}$ may not; pulling back gives obstr. data which is empty ~~...~~ perturbed const. component, ok b/c const. discs are transversely cut out).

Similarly, can write down "consistency conditions" for bundles of moduli space via pullbacks.

The technical conditions on E:

(1) <u>semicontinuity</u>:

If $\mathbb{P} \in \mathcal{M}_{k+2}(\beta)$, &

$q \in \mathcal{M}_{k+1}(\beta)$ is ε close to $\mathbb{P}$ (implicitly fix stabilization data)

Suppose now $\mathbb{X}$ is δ close to q $\Rightarrow$ $\mathbb{X}$ is $\varepsilon + O(\delta)$ close to $\mathbb{P}$

(like a triangle inequality, except careful: this isn't a topology):

Semicontinuity: The data $E_{\mathbb{X}}(\mathbb{P})$ is <u>semicontinuous</u>: If for each $\mathbb{P}$ $\exists$ ε s.t.

if q is ε-close to $\mathbb{P}$, then $E_q(\mathbb{X}) \leq E_{\mathbb{P}}(\mathbb{X})$ for $\mathbb{X}$ suff. close to q.

both in $C^\infty(\Sigma', u'^* TX \otimes \Lambda^{0,1})$ $\quad x \in (\Sigma', z; u')$.

(automatically shows $U_q \subset U_{\mathbb{P}}$)

$\bar{\partial} u' = E_q(u') \subset E_p(u')$)

"<u>coord. change</u> of Kur. chart." gives

& cocycle condition is obvious b/c they are just inclusion maps

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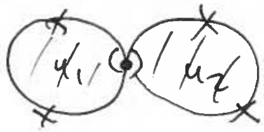
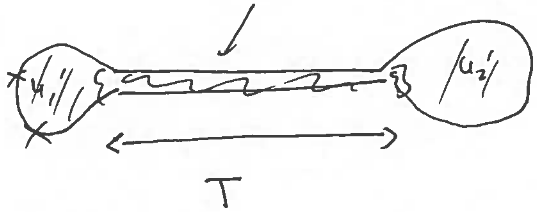
(2) exponential decay.

$\mathbb{P} =$ 

"thin"

$E(x)$ independent of this part.

$E(x)$ depends only on u_1', u_2'

X 

element is supported on thick part

(means: if change map on thin part, $E(x)$ doesn't change).

u_i'  $\longrightarrow (X, L)$

K_i

$u_i' \in L^2_k(K_i', X)$ ~~want~~ this space indep. of T & everything.

$E(x) \subset$ ~~...~~ $L^2_k(K_i', u_1'^* TX \otimes \Lambda^{0,1})$

T $\quad\quad \cong$ fixed functions space

$e_1, \dots, e_N$ basis. $\quad L^2_k(K_i, u^* TX \otimes \Lambda^{0,1})$.

$E: L^2_k(K_1', X) \times L^2_k(K_2', X) \times [T_0, \infty)_T \longrightarrow$ ~~...~~ $L^2_k(K_i, u^* TX \otimes \Lambda^{0,1})$

Also need a relative existence theorem

(given data on boundary of moduli space (of lower strata), can always extend in),

exp. decay in T: $\quad \left|\dfrac{\partial^n e_i}{\partial T}\right|_{C^0} \le C_{n,p}\, e^{-c_n T}$.

& smooth in other parameters.