P. Seidel, <u>Gauss-Manin connections from a TQFT viewpoint</u> 10/10/2016

Ground field $\mathbb{C}$.

$X \xrightarrow{\pi} S$ smooth family of alg. varieties.

S affine (though this is completely unnecessary), $R = \Gamma(S, \mathcal{O}_S)$ commutative $\mathbb{C}$-algebra.

$\rightsquigarrow H^*_{dR}(X/S)$ fibrewise de Rham cohomology (a sheaf on S).

This is the cohomology of the <u>algebraic</u> de Rham complex. ("(polynomial) $\overset{\text{alg.}}{\text{diff'l}}$ forms")

$$H^*_{dR}(X/S) := R\pi_*\left(\Omega^0_{X/S} \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S} \to \cdots\right)$$

(π proper, $S = pt$; easy to see this is usual DR coh. via GAGA; non-proper case: still true, <u>harder</u>).

The Gauß-Manin connection is a canonical $\overset{\text{(flat)}}{\text{connection}}$ ∇ on $H^*_{dR}(X/S)$.

<u>If</u> <u>π is proper</u>, $H^*_{dR}(X/S)$ is a graded coherent sheaf on S, and hence ∇ makes it into a local system. (Thus: $\overset{\pi\text{ proper,}}{\text{having}}$ a connection $\Rightarrow$ v.b., integrable/flat $\Rightarrow$ local. sys.).

Let's look at the special case when X is <u>affine</u>.

Have: $0 \to T(X/S) \to TX \to \pi^* TS \to 0$

$$\begin{tikzcd} H^0(X, TX) \arrow[r] & H^0(X, \pi^*TS) \arrow[l, bend left, "\text{so, } \exists \text{ lift.}"] \arrow[r] & \overset{\overset{0 \text{ because } X \text{ is affine.}}{\parallel}}{H^1(X, T(X/S))} \end{tikzcd}$$

$\uparrow$ in general, this is given by the action of the Kodaira-Spencer class.

Take a vector field σ on S, lift it to a vector field ξ on X; then

$$L_\xi : \Omega^*_{X/S} \to \Omega^*_{X/S} \quad \text{(Lie action)}$$

is compatible with the de Rham diff'l, & induces ∇_ξ.

Why is this canonical? (need to glue affine pieces)!

Answer: Cartan homotopy formula!

Says: if η is a vector field along the fibres, then

$$L_\eta = di_\eta = i_\eta d .$$

Any two lifts ξ differ by such an η, hence L_ξ is canonical up to chain homotopy.
(can use this to globally construct, by gluing together connections associated to lifts on affines)

Change of notation:

Consider

$$(\Omega^{-*}_{X/S}[\![u]\!], \underset{}{\overset{\text{mult. by } u}{u}}\overset{\text{deg } -1 \text{ on } \Omega^{-*}}{d}),$$

where u is a formal variable of degree 2.

$[\![u]\!]$ ← "graded completion"

The cohomology of this sits in a long exact sequence

$$\to H^*(\text{" "}) \xrightarrow{u} H^*(--) \xrightarrow{u=0} R\pi_*(\Omega^{-*}_{X/S}) \to --$$

More concretely, $H^*(-)$ is one copy of <u>closed forms</u> + ∞'ly many copies of $H^*_{dR}(X/S)$.

Instead of a connection, this carries "u times a connection."

Meaning, have

$$\begin{tikzcd}
H^*(\text{" "}) \arrow[r, "\Gamma_\sigma"] \arrow[d, "u=0"] & H^{*+2}(--) \arrow[d, "u=0"] \\
R\pi_*(\Omega^{-*}_{X/S}) \arrow[r, "\substack{\text{Kodaira-}\\\text{Spencer}\\\text{class}}"'] & R\pi_*(\Omega^{-*}_{X/S})
\end{tikzcd}$$

satisfying

$$\Gamma_\sigma(fx) = f\Gamma_\sigma(x) + u(\partial_\sigma f)\Gamma(x).$$

f fcn. on S.

(unusual formulation: usual formulation is Griffiths transversality which says failure of ∇_{GM} to be compatible w/ Hodge filtration is the <u>Kodaira-Spencer class</u>).

In this formulation, the story <u>generalizes</u>.

Getzler-Gauss-Manin connection in noncommutative geometry.

Let A be an associative algebra (<u>free</u> over $R =$ commutative $\Gamma(S, \mathcal{O}_S)$). We can associate to it

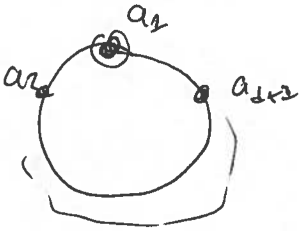
↑ restrictive in alg. geom, but not so much if we replace A by a <u>free</u> dga.

the Hochschild chain complex

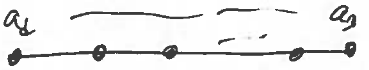
$$CC_{-d}(A) = A^{\otimes d+1} \oplus A^{\otimes d} \quad \text{exists only if } d>0$$

~~The differential~~

Think of generators of the two pieces as

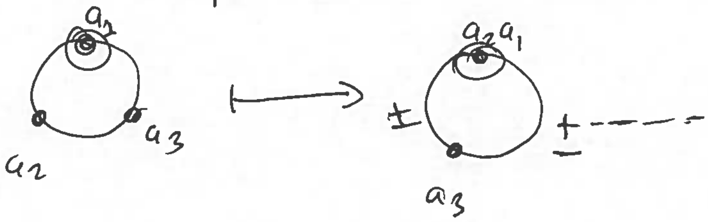


and



The differential multiplies adjacent entries & also closes up the necklace.

ex :



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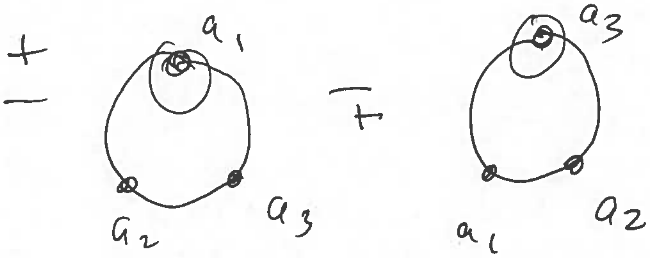
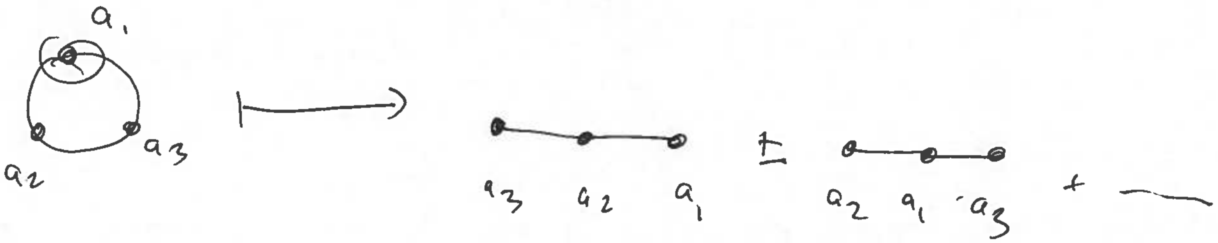


The cohomology

$HH_*(A)$ e.g. $HH_0(A) = A/[A,A]$. If $A = \Gamma(X, \mathcal{O}_X)$ for X affine, then $HH_*(A) \cong R\pi_*(\Omega_{X/S})$ with grading reversed.

There is an additional operation, the Connes B operator

$\Delta : C_*(A) \longrightarrow C_{*-1}(A)$ annihilates open necklaces, &



This satisfies $d\Delta + \Delta d = 0$, $\Delta^2 = 0$; consider

$$\left(C_*(A)[[u]],\ d + u\Delta\right).$$

The cohomology of this called <u>negative cyclic homology</u> $HC_*^-(A)$, ~~&~~ fits into:

$$\cdots \to HC_*^-(A) \to HC^-_{*+2}(A) \to HH_*(A) \to \cdots$$

Take a derivation ∂ of R (= a vector field σ on S, $\partial = \partial_\sigma$).

Lift that to a derivation of A as an R-module (but not an algebra), and hence to a derivation of the Hochschild complex ~~as an R-module.~~ and of $C_*(A)[[u]]$ as an R-module.

(just pick a basis & differentiate).

Denote the outcome by $\not\nabla$. This is <u>not</u> compatible with the algebra structure, or with the differential. See how it fails to be compatible:

Consider: $\lambda = \not\nabla(d+u\Delta) - (d+u\Delta)\not\nabla$ (Rmk: this acts by 0, but is $\boxed{\text{not nullhomotopic!}}$, b/c $\not\nabla$ is <u>not</u> R-linear (it's a <u>derivation</u>!);

which is an endomorphism of $(C^*(A)[[u]], d+u\Delta)$ of degree $\underline{1}$.

The noncommutative Cartan calculus introduces an operation ι^* such that

$$(d+u\Delta)\iota^* - \iota^*(d+u\Delta) = u\lambda. \quad (\lambda \text{ is not nullhtpic, but } u\lambda \text{ is!})$$

The Getzler connection on $HC_*^-(A)$ is induced by

$$\Gamma = u\not\nabla - \iota^*.$$

(when $u=0$, ι is a chain map on $u=0$ thing, & is action of KS class).

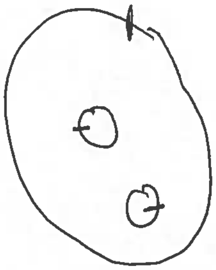
TQFT framework: fix our algebra R and a derivation ∂ ~~and~~ $: R \to R$.

Suppose that we are given a chain complex (C^*, d) of free R-modules. C^* ~~bounded~~ unbounded

(Rmk: R finite global dim, ⇒ flat, ⇒ bounded),

(C^*, d) will carry operations induced by families of surfaces, satisfying axioms...

(framed disks),



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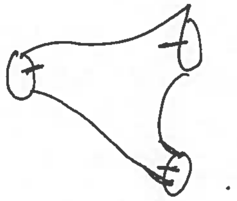
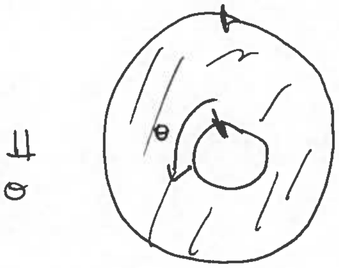


: $C^* \otimes C^* \longrightarrow C^*$.

ex (non-trivial families):



: $C^* \xrightarrow{\Delta} C^{*-1}$

Then, $d\Delta + \Delta d = 0$, $\Delta^2 \simeq 0$ null htpic

(param. by two torus, actually bounded by solid torus).

Can define

$$C^*_{eq} = \left(C^*[[u]]\right)$$

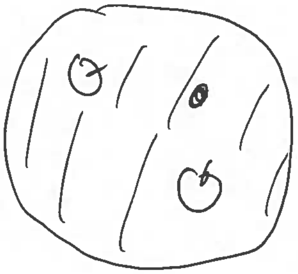
$$d_{eq} = d + u\Delta + \_\_\_$$

higher terms corresponding to this null htpy.

$$\rightsquigarrow H^*_{eq} = H^*(C^*_{eq}, d_{eq}),$$

(want also "identity map" : using this annulus, perhaps).

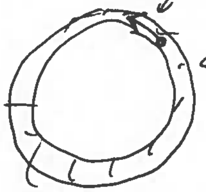
Additional axioms: we allow our surfaces to carry one marked point:



$: C^* \otimes C^* \longrightarrow C^{*+2}$

"Kodaira-Spencer field" (insertion)

Degenerate case:



yields $r: C^* \longrightarrow C^{*+1}$

chain map

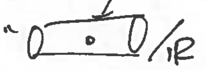
(C^{*-1+2})

Let $\not\nabla$ be a lift of ∂ to a derivation of C^*.

Differentiation axiom(s)

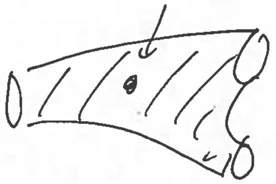
think of d as ""

anywhere

"" =  < on circle

- $\not\nabla d - d \not\nabla = r.$ =

arbitrary position.

- d $\not\nabla$  −  $\not\nabla$ = 

any operation

c.f. "divisor axiom of GW theory"

(point: $r \rightsquigarrow \lambda$ from before.

& $u\, r$ is null htopic; hence can correct $\not\nabla$).

λ, ι, Cartan htopy formula all exist!)

⇝ As a consequence,

H^*_{eq} carries a Getzler-Gauss-Manin connection.

Rmk: • could try to show these axioms are classically satisfied classically or noncommutatively.

• sympl. geometry comes w/ this framework baked in!