N. Sheridan, Symplectic mapping class groups and homological mirror symmetry 10/17/2016

(joint w/ I. Smith) (IAS Members' Seminar)

$X = (X, \omega)$ symp. manifold. (ex: X a cplx m'fold, ω = Kähler).

$$\mathrm{Symp}(X) = \{ \phi : X \underset{\text{diff.}}{\xrightarrow{\cong}} X, \ \phi^*\omega = \omega \}.$$

Ex: $\dim X = 2$, ω = area form $\Rightarrow$ $\mathrm{Symp}(X) \xhookrightarrow{\simeq} \mathrm{Diff}^+(X)$ ← orientation-preserving diffeomorphisms

[Gromov]: • $\mathrm{Symp}^c(\mathbb{R}^4, \omega_{std}) \simeq *$ (but note that $\mathrm{Diff}^c(\mathbb{R}^4)$ is unknown).

• $\mathrm{Symp}(\mathbb{CP}^2, \omega_{FS}) \simeq PU(3)$

• $\mathrm{Symp}(\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_0 \oplus \omega_0) \simeq (SO(3) \times SO(3)) \rtimes \mathbb{Z}/2$

on the other hand,

$$\mathrm{Symp}(\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_0 \oplus \lambda\omega_0) \simeq \begin{cases} (SO(3)\times SO(3)) \rtimes \mathbb{Z}/2 & \lambda = 1 \\ \text{varies widely} & \lambda \neq 1 \\ & \text{(for instance, } H^*(-;\mathbb{Q}) \text{ jumps every time } \lambda \text{ crosses } \mathbb{Z}\text{)} \end{cases}$$

$\pi_0 \mathrm{Symp}(X)$ = symp. mapping class group.

(Rmk: π_0 is w.r.t. what topology on Symp? if X has no H^1, C^∞ topology is sufficient).

One source of interesting classes in $\pi_0 \mathrm{Symp}(X)$:

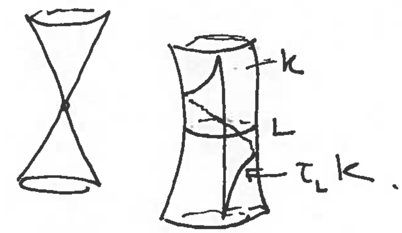
$$\begin{array}{ll} \mathcal{X} \subset \mathbb{CP}^N \times B & \text{family of smooth } \mathbb{C}\text{-proj. varieties} \\ \downarrow & \xrightarrow[\text{induces}]{} \ \pi_1(B, b) \xrightarrow[\text{monodromy}]{} \pi_0 \mathrm{Symp}(X_b, \omega_{FS}|_{X_b}). \\ B & \end{array}$$

E.g., $\overline{B} = \mathbb{D} = \{ z \in \mathbb{C} ; |z| < 1 \}$, $B : \mathbb{D} \setminus 0$, with

X_0 having a node (A_1 singularity)

$$\rightsquigarrow \pi_1(B, b) \longrightarrow \pi_0 \mathrm{Symp}(X_b)$$
$$\begin{array}{ccc} \pi_1(B,b) & & \\ \cong & & \\ \mathbb{Z} & & \\ 1 & \longmapsto & \text{Dehn twist } \tau_L, \text{ where } L = \text{vanishing cycle}. \end{array}$$

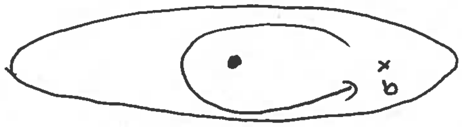
Picture:



on homology,

$$(\tau_L)_* : H_*(X) \to H_*(X)$$
$$a \longmapsto a \pm (a \cdot L) L$$

"Picard-Lefschetz reflection."



When $\dim X = 4$: $(\tau_L^2)_* = id. \Rightarrow \tau_L^2$ lies in $\pi_0 Symp^\circ(X) \subset \pi_0 Symp(X)$ ← those acting trivially on homology.

(τ_L^2 even lies in $\pi_0 Symp^\circ(X)_{triv}$

$$= \ker(\pi_0 Symp(X) \to \pi_0 Diff(X)))$$

[Seidel]: $\exists$ embeddings

specific noncompact manifolds.

braid group $\dashrightarrow$ $B_n \hookrightarrow \pi_0 Symp^c(X)$

$\cup \qquad \cup$

pure braid group $\rightsquigarrow$ $PB_n \hookrightarrow \pi_0 Symp^{\circ,c}(X)_{triv}$.

specific noncompact X

[Keating]: $\mathbb{F}^2 \hookrightarrow \pi_0 Symp^{\circ,c}(X)_{triv}$

(Rmk: $\Sigma_2 \times S^2$ has this too, by foliation methods; but $\pi_1(-) \neq 0$.) but actually lies in $Symp^\circ(X)$, not $Symp^\circ(X)_{triv}$.

Thm (Smith-S.) $\exists$ (X,ω) a compact symplectic 4-mfld such that $\pi_0 Symp^\circ(X,\omega)_{triv.}$ is (countably) infinitely generated, w/ $\pi_1(X) = \{1\}$. (Rmk: $\pi_0 Symp^\circ(X,\omega)_{triv.}$ is a countable group).

(meaning, there's no presentation w/ finitely many generators)

Describe X in a few steps:

$$\bar{\mathcal{X}}''' := \{x_1^4 + x_2^4 + x_3^4 + x_4^4 = 4t\, x_1 x_2 x_3 x_4\} \subset \mathbb{CP}^3 \times \mathbb{C}$$

$\downarrow t$ roots of unity

$\mathbb{C}$ $\qquad G := \ker\left({(\mathbb{Z}/4)^4}/{\mathbb{Z}/4} \xrightarrow{\Sigma} \mathbb{Z}/4 \right) \circlearrowright X_t$.

diagonal

kernel preserves monomial $x_1x_2x_3x_4$.

$$\bar{\mathcal{X}}'' := \bar{\mathcal{X}}'''/G.$$

$\Rightarrow$ each X''_t has 6 A_3 singularities, ($\mathbb{C}^2/(z,w)\sim(iz,-iw)$ locally).

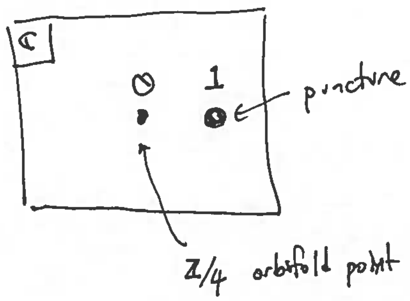
Resolve to get

$$\begin{array}{c} \overline{\mathfrak{X}}' \\ \downarrow t \\ \mathbb{C} \end{array}, \qquad \text{\& note } X'_t \cong X'_{it} \qquad \text{(comes from looking at } G \text{ action on the total space)}$$

$\rightsquigarrow$ get family $\begin{array}{c} \overline{\mathfrak{X}} \\ \downarrow \\ B := \mathbb{C} \text{ with } \mathbb{Z}/4 \text{ orbifold point at } 0 \end{array} \cong \mathbb{C}/(\mathbb{Z}/4)$.

(origin is a fixed point of $t \mapsto it$)

The fibre X_1 has a node, others smooth.

$\Rightarrow$ get $\begin{array}{c} \mathfrak{X} \\ \downarrow \\ B := \end{array}$ 

family of smooth projective variety.

(could compactify B to $\mathbb{P}^1$, but fibre at ∞ would definitely be n.c., not smooth).

Take $E_1, \dots, E_{18}$ = classes of exceptional curves in $H^2(X_b)$ (each singularity resolves into 3 exc. curves)

$E_{19} :=$ pullback of hyperplane section $\in H^2(X_b)$.

We consider Kähler forms with

$[\omega] = \sum_i \lambda_i E_i$ where λ_i rationally independent. (linearly independent over $\mathbb{Q}$)

(Rmk: $\begin{array}{c}\mathfrak{X} \\ \downarrow \\ B\end{array} \overset{\text{can embed}}{\hookrightarrow} \prod_i \mathbb{C}P^{N_i} \times B$, w/ some combination of $(\omega_{FS})_i$'s pulling back to this ω, or sth. near it).

For such ω, $\exists$ monodromy map

"orbifold π_1" $\rightarrow$ $\pi_1^{orb}(B, b) \xrightarrow{i} \pi_0 \, Symp(X_b, \omega)$

(loop going round 0 has order 4 in $\pi_0 Symp$).

$$\begin{array}{ccc} \pi_1^{orb}(B,b) & \xrightarrow{i} & \pi_0 \, Symp(X_b,\omega) \\ \uparrow & & \uparrow \\ \pi_1^{orb}(\tilde{B}, \tilde{b}) & \longrightarrow & \pi_0 \, Symp^0(X_b, \omega)_{triv} \end{array}$$

$\leftarrow$ in fact lies here, by explicit verification!

(cover of B corresp. to subgp = kernel of above map i.)

$\tilde{B} = \mathbb{H} \setminus \{\text{pre-images of } 1 \in B\}$
" upper half plane.

(& monodromy around each pt is a τ_L^2, here we know it's smoothly trivial)

Note that $\pi_1(\tilde{B}) = \mathbb{F}^\infty$.

can think of $\mathbb{H} \to \overline{B}$

Clarify: Covering branched as at $0 \& 1$, ~~?~~ order 4 at 0, order 2 at 1),

now remove pre-image of ~~?~~ 1).

Covering group $= \Gamma_0(2)^+ \subset PSL(2,\mathbb{R})$, where

$$\Gamma_0(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : 2 \mid c \right\}$$

$\Gamma_0(2)^+$ ~~?~~ also has $\begin{bmatrix} 0 & -1/\sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix}$ as an extra element.

__Thm__ (Smith–S.)

a certain subgroup called "Calabi-Yau autoequivalences."

$$\begin{tikzcd}
\mathbb{F}^\infty \cong \pi_1(\tilde{B}, \tilde{b}) \arrow[r] \arrow[d] \arrow[rr, bend left, "\cong"] & \pi_0 \mathrm{Symp}^0(X,\omega)_{triv} \arrow[r] \arrow[d] & \mathrm{Aut}^0_{CY}(\mathrm{Fuk}(X,\omega)) \arrow[d] \\
\pi_1(B,b) \arrow[r] \arrow[rr, bend right, "\cong"'] & \pi_0 \mathrm{Symp}(X,\omega) \arrow[r] & \mathrm{Aut}_{CY}(\mathrm{Fuk}(X,\omega))
\end{tikzcd}$$

$\leftarrow$ Auts which act trivially on Hochschild homology

(CY means: "respects $\mathrm{Hom}^\circ(K,L) \cong \mathrm{Hom}^{2-\circ}(L,K)^\vee$").

__Cor__: $\pi_0 \mathrm{Symp}^0(X,\omega)_{triv} \twoheadrightarrow \mathbb{F}^\infty$.

$\Rightarrow$ infinitely generated.

More precisely, $\pi_0 \mathrm{Symp}^0 \cong \mathbb{F}^\infty \rtimes G$,

$\pi_0 \mathrm{Symp} \cong \pi_1(B) \rtimes G$, where $G = \ker(\pi_0 \mathrm{Symp} \to \mathrm{Aut}(\mathrm{Fuk}))$.

__Proofs__ (next page)

<u>Proof</u>: Use HMS:

$$\begin{tikzcd}
\pi_1(B,b) \arrow[r] \arrow[rr, bend right, "\text{[Bayer-Bridgeland]}"'] & \mathrm{Aut}_{CY}(\mathrm{Fuk}(X,\omega)) \arrow[r, "\cong", "HMS"'] & \mathrm{Aut}_{CY}(D^b\mathrm{Coh}(X^\vee))
\end{tikzcd}$$

"mirror" K3 variety.

(Thm in this case [Smith-S.])

[Bayer-Bridgeland] (when Pic rank of $X^\vee = 1$).

their proof goes via studying <u>stability</u> <u>conditions</u> on $\mathrm{Coh}(X^\vee)$.

(Rmk: this turns out to be answer to all λ_i being <u>rationally</u> <u>independent</u> in $\omega = \sum \lambda_i E_i$).

<u>Rmk</u>: Noether-Lefschetz loci (Pic. rank higher) $\longleftrightarrow$ loci where rational dependencies occur.

<u>Conj</u>: [Bayer-Bridgeland] when Pic $X^\vee > 1$ suggests $\mathrm{Aut}_{CY}(\mathrm{Fuk})$ should <u>jump</u> when $\exists$ rational dependencies.

(Rmk: $\mathrm{Aut}^0_{CY}(\mathrm{Fuk})$ gen. by squared spherical twists, but $\mathrm{Aut}_{CY}(\mathrm{Fuk})$ <u>not</u> gen. by spherical twists b/c (go around orbifold pt, $\mathbb{Z}/4$ <u>symmetry</u>!!)