C. Teleman, The gauged symplectic sigma-model  11/15/2016

$(X, \omega)$ cpct. sympl. mfld

$G \circlearrowright X$ Hamiltonian action, moment map $\mu : X \longrightarrow \mathfrak{g}^*$.  G connected.

$X/\!\!:G = \mu^{-1}(0) : G$ orbifold (quotient)

$$\begin{tikzcd} & X/\!\!:G & \\ G \arrow[r, hook] & X^\circ \arrow[u, "\pi"] \arrow[r, hook] & X \end{tikzcd}$$

fiber of $\pi$  open cell of $|\mu|^2$ Morse Flow

statements:

[Oancea; Ritter – special cases].

$$(1)\quad SH^*(X^\circ) = QH^*(X/\!\!:G;\ \underset{\substack{\uparrow \\ \text{coefficients}}}{\overset{\substack{SH^\circ(T^*G) \\ \shortparallel}}{H_*(LG)}}).$$

Have a (topological) $LG$ action; make $LG$-equivariant

$BLG = G/_{Ad}G$; so can talk about  twisting coefficients

$$SH^*_{LG}(X^\circ) = H^*_G(G; SH^*(X^\circ)) = QH^*(X/\!\!:G).$$

and, } deformation

$$QH^*_{LG}(X) = H^*_G(G; \underline{QH^*(X)}).$$

non-trivial coefficient system.

Conj: deformation. (but trivial 0th order term!)

**Thm**: There is a $G$-equivariant local system over $G$ with fiber at $g \in G = HF^*(X; g)$  Floer hom. of sympl. induced by $G$

$(\cong QH^*(X)$ in connected $G$, cpct. $X$ case!)

the cohomology $H^*_G(G; QH^*)$ is a Frobenius algebra under group convolution. Is the space of states in the gauged GW theory of $X$.

Conj: $H^*_G(G, \underline{QH^*}_{def}) = QH^*(X/\!\!:G)$

→ In the Fano case: no deformation (to first order).

Non-Fano case: $QH^*$ is a summand of LHS. (LHS has "spurious states in gauged model").

from computations in GLSM.

Rmk: If $X = \mathbb{C}^N$, $G = T \hookrightarrow U(1)^N$, Fano case:

Get Batyrev presentation of $QH^*(X/T)$, $= \dfrac{H^*(BT)}{\text{seed relations}}$.

2. A few generalities on Gauged sympl. Sigma-model. for R. surfaces $\Sigma$.

GW theory TQFT: integration via stable hol. maps to $X$

Now, if $G \circlearrowright X$, could

- add a "background field!" $P \to \Sigma$ principle $G$ bdle.
  & look at hol. sections of this, need cplx. structure of $G$-bundle.
- quantizing means integrating over bundles. More than one way to define moduli spaces:
  Salamon-Zilber
  Woodward-Gonzalez.

Favorite notion: use all hol. bdles
- stable sections
- K-theory to integrate

Toy examples: [T.-Woodward] "moduli of a representation of $G$, $X = V$"

finiteness: $X = pt$, $G = GL(1)$, [Frenkel-T.-Tolland.]

[Solis]: defined moduli for nodal curves)

T. w/ Gonzalez

TanGraber.

Have: $(\Sigma, P \to \cancel{X}, \text{section})$

$\downarrow$

polarization choice:

$P$ 1) maps to $\mathrm{Bun}_G(X)$;

which has a line bundle

choice of asymptotics "adiabatic limit"

2) GIT quotient: other extreme of polarization (section in $X^s$)

first extreme: stable bundles, any section

So one extreme upstairs & one downstairs in $X/\!/G$; need to do sth. to go from one to another.

2) Twisted sector problem:

$X$, $G$ as before; say know GW theory of $X$.

What does one need to get gauged theory?

Givental: (algebraic case): Equivariant GW theory.

(case Bundle $P \to X$ is trivial);

e.g. consider $G$ action on moduli space of stable maps; -- $QH^*_G(X)$.

Frobenius algebra over $H^*(BG)$

Turns out not enough information to ~~capture~~ gauge GW theory.

Ex: $G$ finite; over $\mathbb{Q}$ $H^*_G(X)$ just takes invariants

$H^*(X)^G$.

known: This is not the space of states for gauge theory.

Missing: "twisted sectors," one space for each $g \in G$.

(loop space of $X/G$ & loops in $BG$ have monodromy!)

Correct space of states (finite $G$)

$$\Big[\bigoplus_{g\in G} H^*(X^g)\Big]^G$$ invariants under $G$. &1

coh. fixed pt set

("continuous version" is basically $H^*_G(G; \underline{\underline{QH^*}})$.)

Theorem contains the analogue for compact $G$. (except something seems missing b/c doesn't have "fixed pt sets $X^g$.")

Understand this: one step down:

topological Quantum mechanics: $V$, vector space

Partition function $=$ dim $V$.

Gauging it: letting $G$ act on $V$.

Gauge theory: Hilbert space $V^G$.

Partition function is dimension of $V^G = \frac{1}{\# G} \sum_{g\in G} \mathrm{Tr}_V(g)$ "twisted sector formula!"

In 2D: categorification

$* \xrightarrow[\text{before}]{} $ vector space

$\xrightarrow[\text{in 2D}]{}$ category $\mathcal{F}(X)$ Fukaya category

$S^1 \xrightarrow[\text{before}]{} \mathrm{Tr}(\mathrm{Id})$, i.e. dimension

$\xrightarrow[\text{in 2D}]{} HH_*\mathcal{F}(X)$ (ideally $\cong QH^*(X)$).

Rule: everything in characteristic $0$.

Now, need $G$ to act on $\mathcal{F}(X)$

need $HH_*(\mathcal{F}(X)^G)$

Finite case: there is a twisted sector formula, which says:

$$HH_*(\mathcal{F}(X)^G) = \Big[\bigoplus_{g\in G} HH_*(\mathcal{F}(X); g)\Big]^G.$$

Toy example: algebra $A$, $G$ acts by automorphisms, & $HH_0 = A/[A,A]$.

$(A\text{-Mod})^G = \mathrm{Mod}(G \ltimes A)$. And, now check formula (in char 0).

Continue analogy:

$$\text{Reps of } G \underset{\text{language}}{\overset{(*)}{\longleftrightarrow}} \text{vector bundles over } BG.$$

$G$ Lie group: 2 kinds of representations

usual ones (B-model reps.)
topological (A-model reps.):
E.g., $Y$ manifold w/ $G$ action, then $(\Omega^*(Y), d)$ is a topological rep. of $G$.

(here the parts of $(*)$ are useful)

If $G$ connected, action on $H^*(Y)$ is trivial;
to see non-trivial ~~action~~ part of action, form
Borel construction
$$\begin{array}{c} Y_G = \dfrac{EG \times Y}{G} \\ \downarrow \pi \\ BG \end{array} , \; \&$$

consider $R\pi_* \mathbb{C} \in D\mathrm{loc}(BG)$. Triviality $\Rightarrow$ $\underbrace{H^*(Y_G) = H^*(BG) \otimes H^*(Y)}_{\text{usually this }\sim\sim\text{ is just } E_2 \text{ in Leray.}}$

Back in 2D

Hamilt. action of $G$ on $X$,

$g \in G \longrightarrow \Gamma(g_*) \hookrightarrow X \times X^-$ Lag'n induces a functor on the Fukaya category.

$g_0 \rightsquigarrow g_1$ path $\rightarrow$ Ham. isotopy $\longrightarrow$ isomorphism of functors (more like "A model action" can identify action of two elts of differ by path)

$G$ acts on $\mathcal{F}(X)$ via its topology.

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Aut}(\mathcal{F}(X)) \\ \uparrow & & \uparrow \\ \mathcal{P}^*_\lambda G & \longrightarrow & \mathrm{Inn\,Aut}(\mathcal{F}(X)) \\ \uparrow \text{fiber} & & \uparrow \\ \Omega G & \longrightarrow & \mathrm{Aut}(\mathrm{Id}\,\mathcal{F}) \end{array}$$

$\mathrm{Inn\,Aut}(\mathcal{F}(X))$ ← functors w/ an isomorphism to identity (coherently trivialized action of functors)

$\mathrm{Aut}(\mathrm{Id}\,\mathcal{F})$ = units in $QH^*(X)$ or $HH^0(\mathcal{F})$

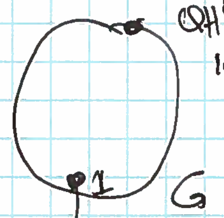
"derived Seidel homomorphism"

Have a $E_2$ ~~map~~ $\boxed{C_* \Omega G \to HCH^*(\mathcal{F}(X))}$ $\left(\overset{\text{if lucky}}{\cong} QH^*(X)\right)$

($E_2$ info contains ~~something~~ ~~large~~ extra structure.)

$E_2$ : $H_* \Omega G \longrightarrow QH^*(X)$ (w/ $\mathbb{C}$-coeffs. pass to $H_*$).

" monodromy map on the local system whose fibers are $QH^*(X)$




$G$-equivariant: capture that by algebra structure of $QH^*_G(X)$ over $H^G_*(\Omega G)$ ← $E_3$ algebra. ($\mathrm{map}(S^2, BG)$) b/c taking the center: of its $E_2$ $HH^*$ of $\Omega G$, which is why $E_3$).

Multiplication on $H^G_*(\Omega G)$ = convolution on $\Omega G$ (this is why $E_2$).

__Facts__: $\mathrm{Spec}\, H^G_*(\Omega G; \mathbb{C}) = BFM(G^\vee)$

[Bezrukavnikov – Finkelberg – Mirkovic]

1) It's a smooth alg. symplectic manifold with symplectic form of degree 2.

(2) If $G = T^n$, its $T^*T^\vee$.

In general, its $\widetilde{(T^*T^\vee/W)}$ ← sympl. affine desingularization.

(3) description in terms of $G^\vee$; comes from 3D/4D gauge theory.

" Coulomb branch of pure 3D $N=4$ gauge theory. "

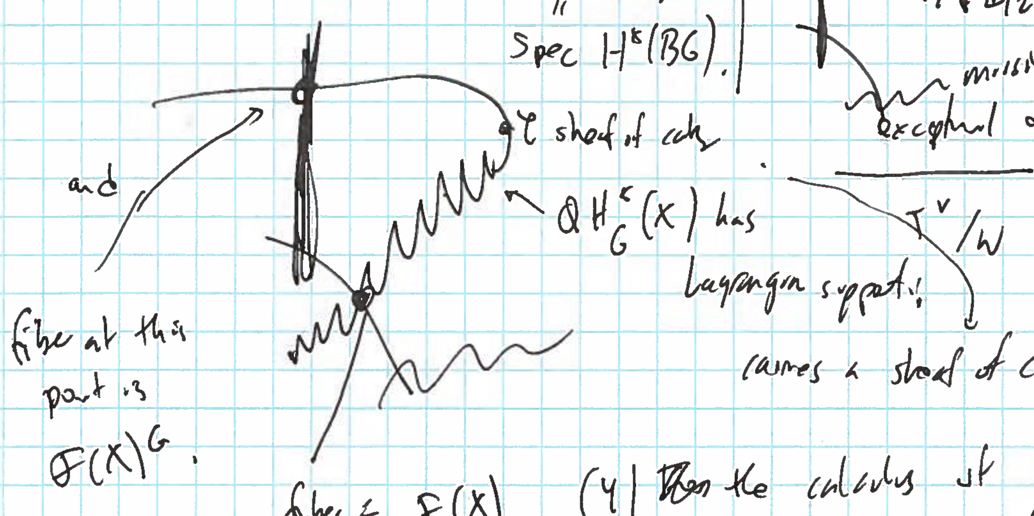
For $SL_2$ — dual is $PSL_2$ (divide by $\mathbb{Z}/2$ + blow up one).

$t/_W = \mathbb{C}$
"
$\mathrm{Spec}\, H^*(BG)$.

$(T, T^\vee)/(\mathbb{Z}/2)$

missing zero section.

exceptional divisor = $\mathrm{Spec}\, H_*(\Omega G)$, Lagrangian inside




(4) Then the calculus of "matrix factorizations" on this space recovers fixed pt. categories recovers Homs between categories, etc.

(5) Equivariant cohomology $H^*_G(G; QH^*(X)) = (QH^*_G) \cap (T, T^\vee)$. (there's actually a "MF-type cat. assoc. to this intersection prob.

All of this is true in the abstract <u>noncommutative dualizable category</u> <u>setting</u>

(sheaf of categories) $\longleftrightarrow$ "Spectral decomposition" . . .