C. Viterbo, Rectification and the Floer complex: quantizing Lagr's in T^*N

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IAS/PU Sympl seminar.

Lagr. submflds (in T^*N) $\longleftrightarrow$ cplxes of sheaves on $N\times\mathbb{R}$

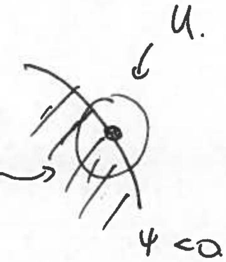
$SS(\mathcal{F}^\bullet) \xleftarrow{\ SS.\ } \mathcal{F}^\bullet$.

<u>Def</u>: Let $\mathcal{F}\in D^b(X)$, $(x_0,p_0)\in T^*X$, $(x_0,p_0)\notin SS(\mathcal{F})$ iff for (a,b) near (x_0,p_0), $\forall\ \psi\in C^\infty$ s.t. $d\psi(x)=p$, $\psi(x_0)=0$

$$\lim_{U} H^\bullet(U,\mathcal{F}) \xrightarrow{\sim} \lim_{U} H^\bullet(U\cap\{\psi<0\},\mathcal{F})$$

<u>Rmk</u>: 1) $SS(\mathcal{F})$ is closed.

2) At section level, for a sheaf, any section defined extends uniquely



<u>Properties</u>:

1) $SS(\mathcal{F})$ is pos. homog.

2) (Kashiwara–Schapira, Gabber) $SS(\mathcal{F})$ is "coisotropic" in some gen. sense, b/c may not be smooth. If, moreover, $\mathcal{F}$ is constructible, then, $SS(\mathcal{F})$ is "Lagrangian"

3) If k_U is the constant sheaf on U, open set w/ smooth boundary, then

$$SS(k_U)=\nu^*U=\left\{(x,p)\ \middle|\ \begin{array}{l} x\in U,\ p=0,\ \text{or} \\ x\in\partial U,\ p=\lambda\nu(x) \\ \qquad\qquad \lambda<0 \end{array}\right\}$$

$\nu(x)$ <u>outward</u> normal at x.

So let $L\subset T^*N$. Set $\hat{L}=\left\{(x,\tau p,t,\tau)\ \middle|\ \begin{array}{l}(x,p)\in L \\ t=f(x,p)\end{array}\quad \tau>0\right\}\subset T^*(N\times\mathbb{R})$

(exact, $\lambda|_L=df_L$ + brane structure)

Then, $\hat{L}$ is pos. homogeneous. We look for $\mathcal{F}_L^\bullet\in D^b(N\times\mathbb{R})$ s.t.

$$SS(\mathcal{F}_L^\bullet)\setminus 0_{N\times\mathbb{R}}=\hat{L}.$$

$\mathcal{F}_L^\bullet$ is called the/a quantization of L. For any exact L, $\exists\ \mathcal{F}_L^\bullet$ quantizes L.

proved by Guillermou, & related to Nadler – Zaslow,

(N cpct.)

<u>Thm</u>: Let $\mathcal{L} \subset T^*N$ be closed, exact, vanishing Maslov class, Spin. Then there exist a ~~unique~~ $\mathcal{F}^\bullet_{\mathcal{L}} \in D^b(N \times \mathbb{R})$ s.t. $SS(\mathcal{F}^\bullet_L) = \hat{\mathcal{L}}$ &

1) $\mathcal{F}_L$ is "pure and simple" (can still twist).

("index" of map $H^\cdot(U, \psi<0; \mathcal{F}) \leftarrow H^\cdot(U, \mathcal{F})$ is always 1. + normalize).

2) $\mathcal{F}_L \simeq 0$ at $t = -\infty$

$\simeq k_N$ at $t = +\infty$.

3) $FH^*(L_0, L_1, a, b) \simeq H^*(N \times [a, b[; R\mathcal{H}om^*(\mathcal{F}_{L_0}, \mathcal{F}_{L_1}))$!

"rel homology of $(-\infty, b)$ rel. to $(-\infty, a)$".

action btw. a and b.

Guillermou

in particular, $FH^*(L_0, \nu^* U; a, b) \simeq H^*(U \times [a, b[; \mathcal{F}_{L_0})$.

(this implies (3) w/ some work.)

4) $\mathcal{F}_L$ is unique provided 1) and 2) are required.

(in particular $\nexists$ local systems on $\mathcal{L}$ which aren't detected by N)

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original approach:

$R\mathcal{H}om^*$ is the adjoint of $*$.

$\mathcal{F}_1 * \mathcal{F}_2$ via:

$\mathcal{F}_1 \in D^b(N \times \mathbb{R})$

$\mathcal{F}_2 \in D^b(N \times \mathbb{R})$

want $\mathcal{F}_1 * \mathcal{F}_2 \in D^b(N \times \mathbb{R})$.

write $s(t_1, t_2) = t_1 + t_2$

$s(x, t_1, (x), t_2) = (x, t_1 + t_2)$

$\mathcal{F}_1 * \mathcal{F}_2 = (R_s)_! \, d_N^{-1} (\mathcal{F}_1 \boxtimes \mathcal{F}_2)$

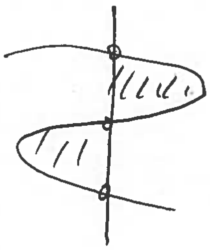
$d_N : N \to N \times N$ diagonal map.

II. Idea for constructing $\mathcal{F}_L$:

One would like $(\mathcal{F}_L)_{(x,t)} \simeq FC^*(L, V, \{t\}) = \lim_{\varepsilon \to 0} FC^*(L, V_x; t-\varepsilon, t+\varepsilon)$

$V_x = x \times (\mathbb{R}^n)^*$ vertical

$(\mathcal{F}_L)_{[x]\times\mathbb{R}} \simeq FC^*(L, V_x)$.



V_x

Possible explanation of Guillermo's proof

More generally, we would like $\mathcal{F}_L^\bullet(U \times]a,b[) = FC^*(L, \nu^* U; a, b)$.

This does not define a presheaf.

If $W \subset V \subset U$, then

$$\begin{tikzcd} FC^*(L, \nu^* U, a, b) \arrow[r] \arrow[dr, dashed] & FC^*(L, \nu^* U, a, b) \arrow[d] \\ & FC^*(L, \nu^* W; a, b) \end{tikzcd}$$

does not commute!

$\mathcal{C}$ = cat. of smooth fcns. on N w/ $\le$.

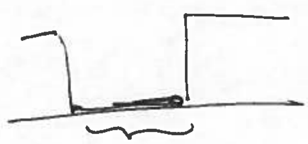
$Mor(f,g) = *$ if $f \le g$

$\emptyset$ otherwise.

Look at, for $f \in \mathcal{C}$ $FC^*(L, df; a, b) = FC^*(L - df; a, b)$

If $f \longrightarrow \infty \cdot \chi_U$

char. fcn.



U

(pick f generic for L & pick a limiting sequence of f's $\to \chi_U \cdot \infty$ all the way)

$FC^*(L, df; a, b) \simeq FC^*(L, \nu^* U, a, b)$

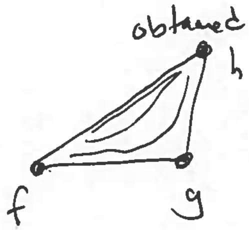
We have, if $f \le g$, get

$FC_L^*(f) = FC^*(L, df)$

(drop a, b); $\mathbb{R}$ part is a sheaf).

& get $FC_L^*(g) \longrightarrow FC_L^*(f)$

obtained by linear homotopy from f to g.



If $\Phi_{(f,g)} : FC_L^*(g) \longrightarrow FC_L^*(f)$

$\Phi_{(f,h)} \neq \Phi_{(g,h)} \circ \Phi_{(f,g)}$, but they are chain homotopic.

What we got is a quasi-presheaf (pseudo-functor) or homotopy-coherent system.

Sketch: 1) Show that $F C^{\infty}_L$ can be rectified to a real presheaf.

2) Sheafify

3) Prove that $SS(\mathcal{F}_L) = \widehat{L}$

4) Draw consequences.

5) generalize.

Rectification: (Lurie, Goerss-Jardine, Vogt, Cordier-Porter, Segal, Dugger)

"htpy limits & colimits" (but does topological category) — Vogt

bnet note — Segal

review paper; covers ch. cplxes case. — Dugger

If $F: \mathcal{C} \to Ch^b$ c-bdded. chain cplxes. "htpy coherent"

then $\underset{(\mathcal{C}\downarrow f)}{\mathrm{hocolim}} = \left(\bigoplus_{f_n \geq \cdots \geq f_1 \geq f} F^\bullet(f_n), D \right)$ is the rectification $\widehat{F}(f)$.

where $d_i(f_n \geq \cdots \geq f_1 \geq f) \otimes x = (f_n \geq \cdots \geq \widehat{f_i} \geq \cdots \geq f_1 \geq f) \otimes x$

$i = n$: $d_n(f_n \geq \cdots \geq f) \otimes x = (f_{n-1} \geq \cdots \geq f) \otimes \Phi_{f_{n-1} f_n}(x)$

$d = \sum (-1)^i d_i$

$\partial: F^q(f) \to F^{q-1}(f)$

$D = d + \partial$.

$\Rightarrow$ rectification.

$\mathrm{Mor}(f, g)$, $f \leq g$, induces $\mathrm{Hocolim}_{(\mathcal{C}\downarrow g)} F^\bullet \longrightarrow \mathrm{Hocolim}_{(\mathcal{C}\downarrow f)} F^\bullet = \widehat{F}(f)$

$(f_n \geq \cdots \geq g) \otimes x \longrightarrow (f_n \geq \cdots \geq f) \otimes x$

& $\widehat{F}^\bullet(f) \to F(f)$ is a ch. htpy equiv.

note:

$\rightsquigarrow$ presheaf, $\rightsquigarrow$ sheaf.

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III. Prove that $SS(\hat{\mathcal{F}}_L)$ is $= \hat{L}$.

Rmk: at generic points, no difficulty.

tricky part is where there is a fold / bifurcation.

difficult: $SS(\hat{\mathcal{F}}_L) \subset \hat{L}$ ($\hat{L} \subset SS(\hat{\mathcal{F}}_L)$ easily by a limiting argument; pass to closure)

~~difficult~~

hard to see

it's not bigger, if construct by hand.

(can use Mayer vietoris to localize this); & use a local generating fcn. for the Lagrangian, to compare).

Prop: If $Z \subset N$, submfld, check:

Then, $FH^*(L, \nu^* Z; a,b) \cong H^*(N \times [a,b[, R\mathcal{H}om^*(k_Z, \mathcal{F}_L))$

ex: apply this to $Z = \Delta_N$, $L = L_0 \times L_1$, & this implies

$\Rightarrow FH^*(L_0, L_1; a,b) = H^*(N \times [a,b[, R\mathcal{H}om^*(\mathcal{F}_{L_1}, \mathcal{F}_{L_2}))$

IV. Appl. & generalize.

$\mathcal{F}$ sheaf on $N \times \mathbb{R}$, $L \subset T^*N$

$FH^*(L, \mathcal{F}) \overset{\text{def}}{=} H^*(N \times \mathbb{R}, R\mathcal{H}om^*(\mathcal{F}_L, \mathcal{F}))$.

- ~~[illegible]~~, can define $FH^*(L_1, L_2)$ for L_1, L_2 non-smooth;

(need uniqueness; but in certain situations, can get this).

- $T^*N = $ "L" coisotropic

"$SS(\mathcal{G})$", $\mathcal{G}$ is a flabby sheaf.

"$FH^*(0_N, T^*N) = H^*(N \times \mathbb{R}, \mathcal{G}) = \mathcal{G}$ in degree 0 (is not unique!)

0 otherwise (but $= 0$ in deg > 0).