IAS conference, 3/13/2017

D. Auroux, Towards HMS for complete intersections in toric varieties

(in progress, w/ M. Abouzaid, since 2012) (B-model on complete intersection)

Setups (Hori-Vafa, — A-Abouzaid-Katzarkov).

- hypersurface in $(\mathbb{C}^*)^n$ : $\{f = \sum_{\substack{\alpha\in A \\ A\subseteq\mathbb{Z}^n \\ \text{finite set}}} c_\alpha t^{\rho(\alpha)} x^\alpha = 0\} = H$, where $|t| \ll 1$ (or Novikov parameter).

$\rho : A \to \mathbb{R}$ (satisfying some convexity condition)

(if in Novikov world, $c_\alpha$ could be a power series in $t$; $c_\alpha$ unitary).

($\rightsquigarrow D^b Coh(H)$)

- In toric variety $V$;

$H = f^{-1}(0)$, $f \in \mathcal{O}(\mathcal{L})$.

* In complete intersection, $H = f_1^{-1}(0) \cap \dots \cap f_k^{-1}(0)$, $f_i \in \mathcal{O}(\mathcal{L}_i)$

(codim $k$ in toric $n$-fold)

→ mirror is a Landau-Ginzburg model $(Y, W)$, where

$Y$: toric CY $(n+k)$-fold, w/ moment polytope

$$\Delta_Y = \{(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_k) \in \mathbb{R}^n \times \mathbb{R}^k \mid \eta_i \geq \varphi_i(\xi_1, \dots, \xi_n)\}, \text{ where}$$

$$\varphi_i = \mathrm{Trop}\, f_i = \max_{\alpha\in A}(\langle\alpha,\xi\rangle - \rho(\alpha)),$$

& $$W = \sum_{i=1}^k w_i^0 + \sum_{\nu:\text{rays of fan of } V} k_\nu T^{c(\nu)} z^{\langle\nu,\alpha_{max}(\nu)\rangle} + (\text{——})$$

extra things which seem to be irrelevant for this side of HMS

(rather: $\nu$ gen. ray of fan of $V$).

↑

(shouldn't ~~affe~~ change crit locus b/c coeffs, Novikov & convex hull of the coeffs),

$$w_i^0 = -z^{(0,\dots,0,1,0,\dots,0)}$$

↖ ($\eta_i$ coordinate)

$$W_\nu^\nabla = k_\nu T^{c(\nu)} z^{\langle\nu,\alpha_{max}(\nu)\rangle}$$

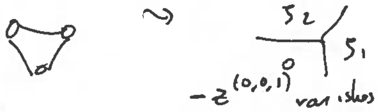
$\rightsquigarrow \boxed{(\mathcal{F}(W))}$

key example: $(n-1)$-pair of pants

$$H = \{x_1 + \dots + x_n + 1 = 0\} \subset (\mathbb{C}^*)^n.$$

$\rightsquigarrow$ mirror: $Y = \mathbb{C}^{n+1}$, $W = -z_1 - \dots - z_{n+1}$

$\{x_1+x_2+1=0\}$ $\rightsquigarrow$ $\varphi = \max(\xi_1, \xi_2, 0)$



$(\simeq \mathbb{C}^3)$.

$-z^{(0,0,1)}$ vanishes to order 1 on each facet, so $\simeq$ product of coeffs.)

(or compactify to $\mathbb{P}^{n-1} \subset \mathbb{P}^n$; $Y$ same, $W = -z_1 - \dots - z_{n+1} + Tz_1 + \dots + \overline{T}z_{n+1}$).

(2d stabilization of usual mirror of $\mathbb{P}^{n-1}$).

Statement:

1) can define a "fibrewise wrapped" "fiberwise admissible" Fukaya category $\mathcal{F}(Y,W)$

(the working def'n ~~[illegible]~~ is ad hoc & uses toric structure of $Y$.)

[hope: Z, Sylvan, or Ganatra-Pardon-Shende might give more general def'ns]

2) can construct objects $L_{\mathcal{L}}$ of $\mathcal{F}(Y,W)$ for all $\mathcal{L} \in \mathrm{Pic}(V)$

(conj: they generate; but this conj. would be more sensible once def'ns ~~[illegible]~~ made more general)

3) can calculate

$$H^* \mathrm{Hom}(L_{\mathcal{L}}, L_{\mathcal{L}'}) \cong \mathrm{Ext}^*_{D^b(H)}(\mathcal{L}|_H, \mathcal{L}'|_H).$$

w/ composition structure matching.

[want: $A_\infty$ structure matching].

After $A_\infty$ enhancing, would get:

$\{$ subcat of $D^b\mathrm{Coh}(H)$ given by line bdles restricted from $V \} \xleftrightarrow{\sim} \{$ subcat of $\mathcal{F}(Y,W)$ consisting of $L_{\mathcal{L}} \}$

& hope these are enough to generate both sides

(known for $D^b\mathrm{Coh}(H)$ side).

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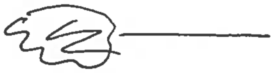
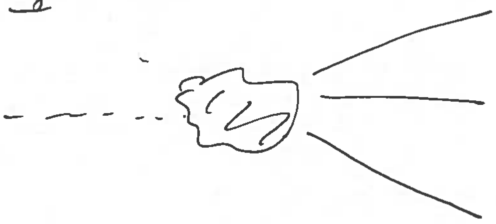
① $\mathcal{F}(Y,W)$?

conventional wisdom: $W(L)$ should look like .

changes: allow



multiple angular sectors, (avoid $-\infty$).

(*: just for simplicity)

↓

objects: properly embedded Lagn $L \subset Y$ not bounding any hol. disks* or unobstructed (poss. w/ bounding co-chains),

such that

Image of $L$ under each of $w_i^0$ is, outside of a cpct. subset, a union of radial straight lines $e^{i\theta}\mathbb{R}_+$, $\theta \neq \pi$.

Fiber of $\vec{w}^0 = (w_1^0, \ldots, w_k^0) : Y \longrightarrow \mathbb{C}^k$ over a point of $(\mathbb{C}^*)^k$

looks like $\approx (\mathbb{C}^*)^n$.. [e.g., pants, $(\mathbb{C}^{n+1}, -z_0 \cdots z_n)$; not general fiber $\approx (\mathbb{C}^*)^n$].

& over toric strata, its a toric degeneration of $(\mathbb{C}^*)^n$. (+ a union of affine spaces), [e.g., fiber of 0 of )].

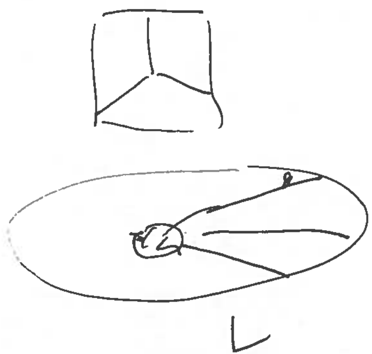
- within fibers, "asymptotic flatness";

$\exists$ a finite collection of toric monomials $z^\lambda$ on $Y$, and an open cover $U_\lambda$ of $\mathbb{R}^n$ (at $\infty$), s.t. for fixed $(w_1^0, \dots, w_k^0)$, as $z \to \infty$ on $L$, with $\log|z| \in U_\lambda$, $\arg(z^\lambda) = c_\lambda$, $(*)$

$\lambda$ includes rays of fan of $V_*$ (if chose $c_\lambda = 0$, then $z^\lambda$ should be real pos. outside cpct. subset).

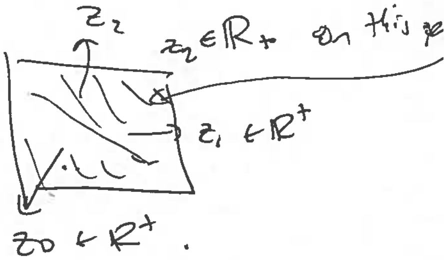
Ex: for $(\mathbb{C}^{n+1}, -z_0 \cdots z_n)$, can ask: for fixed $W$, as $z \to \infty$ whenever $|z_j| > |w|^{1/(n+1)}$, $\arg(z_j) = 0$ on $L$. )

$|2w|^{1/(n+1)}$



$$L \cap \text{fiber} = \ell \subset (\mathbb{C}^*)^2 \xrightarrow{\text{Log}} \mathbb{R}^2$$

$$\| \quad \{z_0 z_1 z_2 = \text{const}\}$$



n.b. (if impose strict equality $\arg(z^\lambda) = c_\lambda$, cannot use std. kähler form on $\mathbb{C}^{n+1}$! b/c parallel transport only preserves this in a limiting sense)

(So instead, define std. kähler form using things like "convex smoothings of max fcn." to get strict equality for $(*)$).

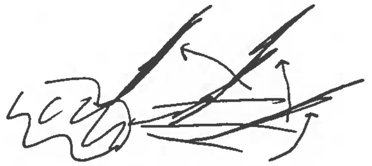
ex: compactify  to , get $-z_0 z_1 z_2 + \underbrace{T_0 z_0 + T_1 z_1 + T_2 z_2}$

so $L$ admissible for $-z_0 z_1 z_2$ $\Rightarrow$ admissible for this monomial too!

$\Rightarrow$ Fibrewise in fibers of $\vec{w}_0 : Y \to \mathbb{C}^k$, $L$ is admissible for extra terms $W_\vee$.

- $\rho_t$: flow in each $\mathbb{C}$ base of $\vec{w}^0$.

moves radial lines ccw without crossing $\mathbb{R}_-$



. lift $\rightsquigarrow$ $\rho_t L$ Lagrangian.

- fibrewise: $\phi_t \in \mathrm{Ham}$, preserves fibres of $\vec{w}^0$

  wraps at constant pace in fibers.

- Define $L^t = \phi_p^t \rho^t L$. (in either order), and

$$\mathrm{Hom}(L, L') = \lim_{t\to\infty} CF(L^t, L')$$ ← this is well defined for all

(assumptions made ensure that, due to behavior at $\infty$ being controlled,

but discrete set of $t$).

Rmk: in fiber of $\vec{w}^0 : Y \longrightarrow \mathbb{C}^k$, have $((\mathbb{C}^*)^n, W_V)$ is exactly the toric mirror to $V$ and

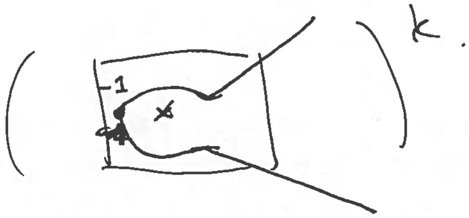
$$\phi^t = \begin{cases} \text{when } V \text{ is compact: "Fukaya-Seidel" category} \\ \text{when } V = (\mathbb{C}^*)^n: \text{ linear wrapped.} \end{cases}$$

At $\vec{w}^0 = (-1, --, -1)$, this is canonical.

Now, given a line bundle $\mathcal{L} \longrightarrow V$, Abouzaid's thesis gives $\ell(\mathcal{L}) \subset (\mathbb{C}^*)^n$ (graph), graph over $\mathbb{R}^n$.

$$\begin{tikzcd} \ell(\mathcal{L}) \arrow[r, phantom, "\subset"] \arrow[dr, "2"'] & (\mathbb{C}^*)^n \arrow[d, "\text{moment map}"] \\ & \mathbb{R}^n \end{tikzcd}$$

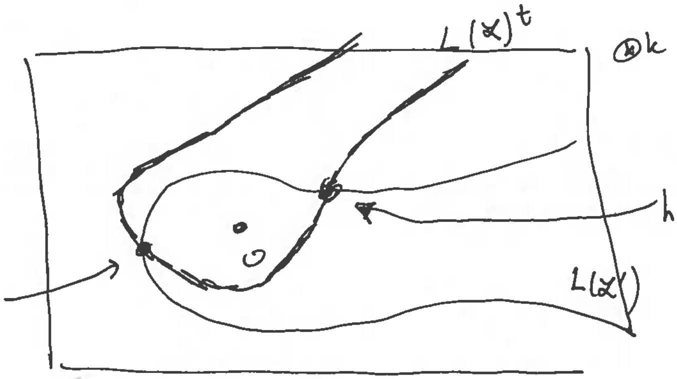
Parallel transport $\ell(\mathcal{L})$ over



(check: fibrewise admissibility preserved).

⤳ get $L(\mathcal{L})$

$\hom(L(\mathcal{L}), L(\mathcal{L}')) = \lim CF(L(\mathcal{L})^t, L(\mathcal{L}'))$. For a large $t \gg 0$, this looks like:



$\otimes k$ ← $2^k$ fibers of $\vec{w}^0$ where intersections happen.

$\mathcal{L}_I = \bigotimes_{i \in I \subset \{1, -, k\}} \mathcal{L}_i$

here, have $\ell(\mathcal{L} \otimes \mathcal{L}_I, \mathcal{L}')$ $\mathcal{L}_i$: $i$th defining line bundle for H

(calculate/checks monodromy ⟷ $-\otimes \mathcal{L}_i$)

$((\mathbb{C}^*)^n, W_V)$

$\cup$

$\ell(\mathcal{L}), \ell(\mathcal{L}')$

Abouzaid's thesis: $CF^*(\ell(\mathcal{L}), \ell(\mathcal{L}')) \simeq \mathrm{Ext}^*_{D^b(V)}(\mathcal{L}, \mathcal{L}')$ (more precisely, Čech model cpx computing Ext).

In general, the $2^k$ pieces, ~~[illegible]~~ are given by, for each $I \subset \{1, \dots, k\}$, by

$$\hom_{(\mathbb{C}^{\times n}, w_V)}\big( \ell(\mathcal{X} \otimes \mathcal{X}_I), \ \ell(\mathcal{X}') \big). \qquad \mathcal{X}_I = \bigotimes_{i \in I} \mathcal{X}_i .$$

- connecting differential: multiplication by $f_i \in \mathcal{O}(\mathcal{X}_i)$'s. (big Koszul complex.
- product easier to understand than differentials.

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