IAS conference, 3/13/2017

T. Dyckerhoff, Topological Fukaya categories with coefficients

in progress, w/ M. Kapranov, V. Schechtman, Y. Soibelman

X Riemann surface (possibly non-cpct, no boundary), $S \subseteq X$ special (or "singular" for the sheaf) points

$$S \xhookrightarrow{i} X \xhookleftarrow{j} X \setminus S$$

$D^b_c(X)$ constructible derived category.

Def: An object $E^\bullet \in D^b_c(X)$ is called a perverse sheaf if

(1) $j^* E^\bullet \simeq L[1]$ L local system on $X \setminus S$.

(2) $H^n(i^* E^\bullet) \cong 0$ $n > 0$

(3) $H^n(i^! E^\bullet) \cong$ $n < 0$. (← cpct. support stalks)

Rmk: Given a t-structure on open & closed, can glue to get a new t-structure. If one glues usual t-str. on S w/ shifted t-str part on $X \setminus S$, get a new t-str. & perverse sheaves are heart of this t-str.

$PS(X, S)$ category of perverse sheaves

- heart of a certain t-structure as in remark ⇝ abelian category
- stable under Verdier duality. (singles out this particular choice of glued t-structure)

Examples:

$$PS(\mathbb{C}, \{0\}) \xrightarrow{\simeq} \left\{ \Psi \underset{b}{\overset{a}{\rightleftarrows}} \Phi \quad \Psi, \Phi \text{ vector spaces, s.t. } \begin{array}{l} T_\Psi := id_\Psi - ba \\ T_\Phi := id_\Phi - ab \end{array} \text{ should both be isomorphisms} \right\}$$

Ψ "nearby cycles", Φ "van. cycles"; "canonical map" & "variation map"; T_Ψ, T_Φ: "monodromy transformations" on both these complexes.

reformulated in terms of stable ∞-categories

Def: (Anno): An adjunction $f : \mathcal{C} \longleftrightarrow \mathcal{D} : g$ (Rmk: sloppy, more data to remember than f, g in adjunction) of stable ∞-categories is called spherical if

$$T_{\mathcal{C}} := \mathrm{Cone}(id_{\mathcal{C}} \xrightarrow{u} g\circ f)[-1]$$

$$T_{\mathcal{D}} := \mathrm{Cone}(f\circ g \xrightarrow{c} id_{\mathcal{D}})$$

are autoequivalences. (u unit, c counit)

Goal: Introduce a category "perverse Schobers": (still X Riemann surface for today).

$PS^{(2)}(X, S)$ category of "perverse sheaves of stable ∞-categories" (reminiscent of Toën's "$K^{(2)}(-)$")

such that

$$PS^{(2)}(\mathbb{C}, \{0\}) \xrightarrow{\simeq} \{\infty\text{-cat. of spherical adjunctions}\}.$$

strategy: circumvent usual def'n of perverse sheaf to not make reference to $D^b_c(X)$ e.g., to not use usual notion of sheaf, then categorify more simply.

[MacPherson: perverse sheaves should be "easier objects" than sheaves.]

Notation:

- (U, U') pair of opens $U' \subseteq U \subseteq X$, $E^\bullet \in D^b_c(X)$, then

$$R\Gamma(U, U'; E^\bullet) := \text{fiber}\left(R\Gamma(U; E^\bullet) \longrightarrow R\Gamma(U'; E^\bullet)\right)$$

"sections of U w/ support away from U'."

- A pair (U, U') of opens is called <u>basic</u> if
  - (1) U disk, U' non-empty disjoint finite union of disks
  - (2) $U \setminus U'$ contractible & $\#|(U \setminus U' \cap S)| \leq 1$.

visual:

Example: for every $n \geq 0$:

$$(\text{Re } z^{n+1} > -1, \ \text{Re } z^{n+1} > 1).$$

Fact: If $E^\bullet$ perverse sheaf & (U, U') basic pair

$$\Longrightarrow R\Gamma(U, U'; E^\bullet) \simeq R^0\Gamma(U, U'; E^\bullet)$$ pure of degree 0. (vector space, <u>not</u> a cplx.)

(calculate w/ LES, or general thm that vanishing cycles of a perverse sheaf are <u>perverse</u>)

Example: (Given $(\mathbb{C}, \{0\})$, $E \in PS(\mathbb{C}, \{0\})$ get $R^0\Gamma($  $; E^\bullet) = \Phi$, & $R^0\Gamma($  $; E^\bullet) = \Psi$.)

<u>Philosophy</u>: perverse sheaves are better measured in terms of their ~~vanishing cycles~~ "relative stalks", e.g., vanishing cycles.

Thm: The functor

$$PS(X, S) \longrightarrow \text{Fun}(\mathcal{B}(X, S)^{op}, \text{Vect})$$

$$E^\bullet \longmapsto R\Gamma(-; E^\bullet)$$

(poset of <u>basic pairs</u> of opens)

is exact and fully faithful, with essential image consisting of those presheaves $\mathcal{F}: \mathcal{B}(X, S)^{op} \longrightarrow \text{Vect}$ satisfying:

(1) [normalization] : $\mathcal{F}\Big($  $\Big) \simeq 0.$

↖ no special point in here

(2) [homotopy invariance]: Given a $(U, U') \subset (V, V')$ so that

weak equivalences of pairs
[(1) $\pi_0(U') \xrightarrow{\simeq} \pi_0(V')$ ($\Rightarrow$ weak h.topy equiv. of pairs)
(2) $(U \setminus U') \cap S = (V \setminus V') \cap S$,

then $\mathcal{F}(V, V') \longrightarrow \mathcal{F}(U, U')$ is an isomorphism.

(3) [descent]: Given basic pair (U, U') & $U = U_1 \cup U_2$ admissible cover,
have a pullback square

$$\begin{array}{ccc} \mathcal{F}(U, U') & \longrightarrow & \mathcal{F}(U_1, U_1 \cap U') \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{F}(U_2, U_2 \cap U') & \longrightarrow & \mathcal{F}(U_1 \cap U_2, U_1 \cap U_2 \cap U') \end{array}$$

admissible: provided these are all basic pairs.

([warning]: basic pairs don't define a Grothendieck topology, making certain arguments rather delicate, for instance, doesn't contain all intersections / fiber products.)

Relation to example of $(\mathbb{C}, \{0\})$:

satisfying (1)–(3)

~~Fun(B(...))~~ $\text{Fun}(\mathcal{B}(\mathbb{C}, \{0\})^{op}; \text{Vect})^{(1)-(3)}$

↓ • restrict to $\{$  $\}$ ← pairs like this ; (equiv by descent & normalization)
• localize along weak equivalences

"paracyclic category" → $\text{Fun}(\Lambda_\infty^{op}, \text{Vect})^{Seg}$ ↖ sats

↓ not an equivalence, restrict to $\Delta^{op} \subset \Lambda_\infty^{op}$.

$\text{Fun}(\Delta^{op}, \text{Vect})$

Λ_∞^{op} : objects: $\langle n \rangle$ for $n \geq 0$ $\langle n \rangle \longleftrightarrow$ [] n disks .

morphisms: $\langle m \rangle \longrightarrow \langle n \rangle$ are

$\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}$ order preserving satisfying

(*) $\varphi(i + m + 1) = \varphi(i) + n + 1$

"Galois cover" of usual cyclic cat. where $\langle n \rangle$ has automorphism $\mathbb{Z}/n+1$

Δ ↗ morphisms

morphisms sending $[0,m] \to [0,n]$.

(start w/ Δ, & add aut. $\mathbb{Z}$ to each object.

Now

$$\mathrm{Fun}(\Delta^{op}, \mathrm{Vect})^{seg} \longleftrightarrow \mathrm{Ch}_{[0,1]}(\mathrm{Vect})$$

DK-correspondence

$$\mathrm{Ch}_{[0,1]}(\mathrm{Vect}) = \{\Psi \xrightarrow{a} \Phi\}.$$

back to original picture:

$$\mathrm{Fun}(\mathbb{B}(\mathbb{C}, \{0\})^{op}, \mathrm{Vect})^{(1)-(3)}$$

$\downarrow$: steps as on prev. page.

$$\left(\mathbb{Z}[t, t^{-1}, \varepsilon],\ |\varepsilon| = 1\right), \quad \boxed{d\varepsilon = t - 1}$$

$$= C_*(\mathbb{R})$$

(objects w/ $C_*(\mathbb{R})$ action)

$$\mathrm{Fun}(\Lambda_\infty^{op}, \mathrm{Vect})^{seg} \longleftrightarrow \mathrm{Ch}_{[0,1]}(\mathrm{Vect}) = \{\Psi \underset{b}{\overset{a}{\rightleftarrows}} \Phi\}$$

paracyclic DK-corresp.

$T_\Psi \circlearrowleft \Psi$, $T_\Phi \circlearrowleft \Phi$: action of t; b: action of ε.

point: (same way cyclic sets have S^1 action, paracyclic sets have $\mathbb{R}$ action!)

$\downarrow \qquad\qquad \downarrow$

$$\mathrm{Fun}(\Delta^{op}, \mathrm{Vect})^{seg} \longleftrightarrow \mathrm{Ch}_{[0,1]}(\mathrm{Vect})$$

DK-corresp.

$$\mathrm{Ch}_{[0,1]}(\mathrm{Vect}) = \{\psi \xrightarrow{a} \phi\}$$

<u>Def</u>: A <u>perverse schober</u> is a presheaf

$$\mathcal{F} : \mathbb{B}(X, S)^{op} \longrightarrow \{\text{stable } \infty\text{-categories}\}$$

(or A_∞ or dg cat. etc.)

(be careful: need $\mathcal{F} : N(\mathbb{B}(X,S)^{op}) \longrightarrow (\quad)$ in order to build in htpy coherence in defns)

Satisfying

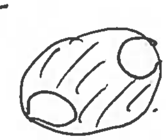
(1) <u>Normalization</u> — as before

(2) <u>homotopy invariance</u> — as before

(3) <u>descent</u>: Given (U, U') & $U = U_1 \cup U_2$ admissible

$$\begin{array}{ccc} \mathcal{F}(U, U') & \longrightarrow & \mathcal{F}(U_1, U_1') \\ \downarrow & \lrcorner\ \text{pullback.} & \downarrow \\ \mathcal{F}(U_2, U_2') & \longrightarrow & \mathcal{F}(U_1 \cap U_2, (U_1 \cap U_2)') \end{array}$$

<u>only if</u>



(4) [codescent]: Given (U, U'), $U' \subseteq Z_1 \cup Z_2 \subseteq U$.

& $(U, Z_1 \cup Z_2)$

$\parallel$

$$\begin{array}{ccc} \mathcal{F}(U, U') & \longleftarrow & \mathcal{F}(U, U' \cup Z_1) \\ \uparrow \ \lrcorner \text{ pushout} & & \uparrow \\ \mathcal{F}(U, U' \cup Z_2) & \longleftarrow & \mathcal{F}(U, Z_1 \cup Z_2) \end{array}$$

(5) [Recollement] Given (U, U') & $V \subseteq U$: have

$$\mathcal{F}(U, U' \cup V) \rightleftarrows \mathcal{F}(U, U') \rightleftarrows \mathcal{F}(V, U' \cap V)$$

$\mathcal{F}\uparrow$ (using homotopy invariance),

gives a recollement of stable ∞-categories,

(Rmk: (4) makes this stable up to Verdier duality)

(5) $\Rightarrow$ Segal up to extension, accounts for failure of descent in more general setting).

Have now:

$$\mathrm{Fun}(\mathcal{B}(\mathbb{C}, \{0\})^{op}, \mathrm{Cat}_\infty^{st})^{(1)-(5)}$$

$\downarrow$ as before

$\mathrm{Fun}(\Delta_\infty^{op}, \mathrm{Cat}_\infty^{st})^{(3),(5)}$ $\longleftrightarrow$ (spherical $S_\bullet$-construction) $(\simeq ?)$ $\{f : \mathcal{C} \to \mathcal{D}$ spherical$\}$

in progress

(tricky/hard b/c of coherence issues!) Seems to need full framework of ∞ categories

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

$\mathrm{Fun}(\Delta^{op}, \mathrm{Cat}_\infty^{st})^{(3),(5)}$ $\longleftrightarrow$ $(\simeq ?)$ $\{f : \mathcal{C} \to \mathcal{D}\}$

relative $S_\bullet$ construction [Waldhausen].

("categorified DK")

Features: • Can give an explicit description

$$L\,\mathcal{B}(X,S)^{op} \xrightarrow{\simeq} \Lambda_{(X,S)} \text{ paracyclic category of } (X,S)$$

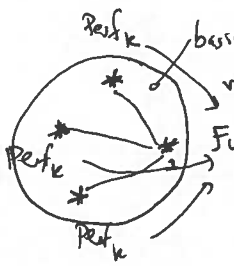
& - $\Lambda_{(X,S)}$ self-dual $\rightsquigarrow$ Verdier duality

- $\Lambda_{(X,S)}$ satisfies van Kampen thm: $\rightsquigarrow$ construct perverse Schobers locally & glue..

(Rmk: relative version of Lurie's Ran spaces & chiral homology, where disks forced to lie inside small disk & touch boundary).

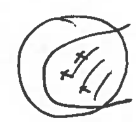
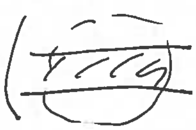
• Can extend perverse Schobers globally via Kan extension (non-trivial: still satisfies descent).

Example: $\pi : X \longrightarrow D$ exact Lefschetz fibration.



[illegible] have spherical functors $\mathrm{van}_i : \mathrm{Perf}_k \longrightarrow \mathrm{Fuk}(\pi^{-1}(1))$

$\overset{\text{using extension}}{\rightsquigarrow}$ defines perverse schober [illegible] $\mathcal{E}_\pi$, satisfying

• $\mathcal{F}($  $; \mathcal{E}_\pi) \longrightarrow \mathcal{F}($  $; \mathcal{E}_\pi)$

first term $= FS(\pi)$ Fukaya-Seidel

second term $= \mathrm{Fuk}(\pi^{-1}(1))$

kernel: $\mathcal{F}($  $; \mathcal{E}_\pi)$ " = " $\mathrm{Fuk}(X)$

Rmk: this sequence is only left exact, would need to "derive" this functor, (& work w/ complexes of categories) somehow producing $\longrightarrow$?

can $\mathrm{Fuk}(-)$ be defined generally as a complex of stable ∞-categories, which can be "derived"?

Next step: higher dimensions. (use similar Milnor stalks as testing objects).

(Rmk: data of Φ, Ψ non-canonical, b/c need to choose (x) direction;) this framework is canonical.