X. Jin, Brane structures from the perspective of microlocal geometry — (joint w/ D. Treumann)

IAS conference, 3/13/2017

Microlocal sheaf theory $\longleftrightarrow$ Lag'n intersection theory.

Nadler-Zaslow: $Fuk(T^*X) \simeq Sh(X)$

Tamarkin: Microlocal category on a compact symplectic manifold, (expected equivalent to Fukaya category.

In this talk, present a way to understand brane structures, & give a top. approach to N-Z; can be generalized over $\mathbb{K}$ = any ring spectrum. (e.g., J-homomorphism appears in one of our theorems).

Part I: Introduction of brane structures & μlocal sheaf theory.

Brane structures: $CF^*(L_0, L_1)$. Say $L_i \subset T^\infty \mathbb{R}^n$. Since linear case, $\exists$ Gauss map

$$L \xrightarrow{\text{Gauss map}} Lag\,Gr(n) := U(n)/O(n)$$
$$x \longmapsto T_x L$$

$\exists$ canon: $U(n)/O(n) \xrightarrow{\det^2} U(1) \simeq B\mathbb{Z}$.

There are two coh. classes of L that are the brane obstructions.

1) Maslov class $= \det^2 \circ$ Gauss $\in [L, B\mathbb{Z}] = H^1(L; \mathbb{Z})$

(& brane str. is a choice of lift of $L \to S^1 = U(1)$ to $\mathbb{R}$)

2)
$$\begin{tikzcd} & & BPin^+(n) \arrow[d] & \\ L \arrow[r] \arrow[urr, dashed, "(*)"] & U(n)/O(n) \arrow[r] & BO(n) \arrow[r, "w_2"'] & B^2\mathbb{Z}/2 \end{tikzcd}$$

(w_2 ~ obstr to lifting)

$\in H^2(L; \mathbb{Z}/2)$

& the Brane str. is a choice of lift $(*)$

One can define $CF^*(L_0, L_1)$ (def'n using analysis, but result is q.iso. inv't under Ham. isotopies so has a topological nature).

$\wr$

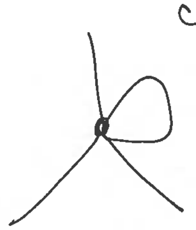
$Hom(\mathcal{F}_0, \mathcal{F}_1)$

$\in Sh(X) \leftarrow \mathbb{R}^n$?

Microlocal sheaves (cont'd).

constructible sheaves on M (= X from before).

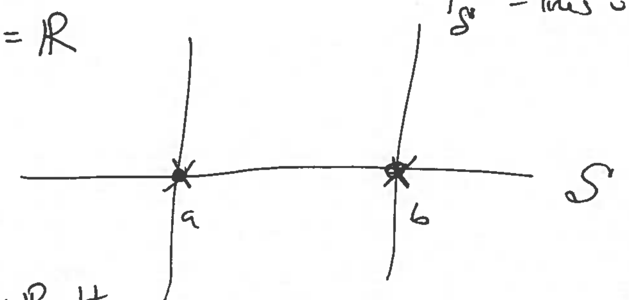
$\mathcal{S} = \{S_\alpha\}$ stratification. $\rightsquigarrow Sh_{\mathcal{S}}(X)$ ← sheaves whose restr to S_α are loc constant



Can also form $T^*_{\mathcal{S}} M = \bigcup_\alpha T^*_{S_\alpha} M$, gives a closed conic Lagrangian in T^*M.

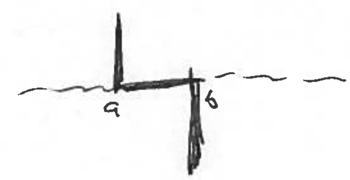
& For each $\mathcal{F} \in Sh_{\mathcal{S}}(M)$, can define $SS(\mathcal{F})$ a closed conic Lag'n $\subset T^*_{\mathcal{S}} M$,

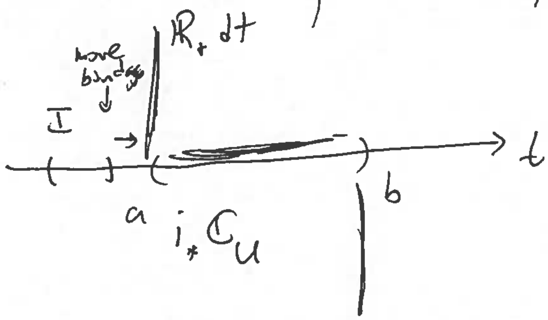
e.g., $M = \mathbb{R}$ $\quad T^*_{\mathcal{S}}$ = lines ∪ zero section.



look at $\mathcal{F} = i_* \mathbb{C}_U$ $\quad U =$ 

then, $SS(\mathcal{F}) =$





$(a, \mathbb{R}_+ dt) \subset SS(\mathcal{F})$ means if move ∂I in (co)direction of $\mathbb{R}_+ dt$, rank jumps.

A central question in microlocal sheaf theory is: given $\Lambda^{conic} \subset T^*M$, find $Sh_{\Lambda^{conic}}(M) \ni \mathcal{F}$

$$Sh_{\Lambda^{conic}}(M) \subset Sh(M), \qquad \mathcal{F} \Leftrightarrow SS(\mathcal{F}) \subset \Lambda^{conic}$$

Usually, one only remembers the ∞-end of $\Lambda^{conic} \subset T^\infty M \cong ST^*M$.

Will denote such a Legendrian by Λ.

Then, one can form the following categories

(a) $Sh_{Cone(\Lambda) \cup T^*_M M}(M) =: Sh_\Lambda(M)$. ($T^*_M M$ ← zero section)

(b) $Sh_{\Lambda}(M) / Loc(M)$ ⟵ invariant of Λ.

↑ SS lies on zero section $T^*_M M$

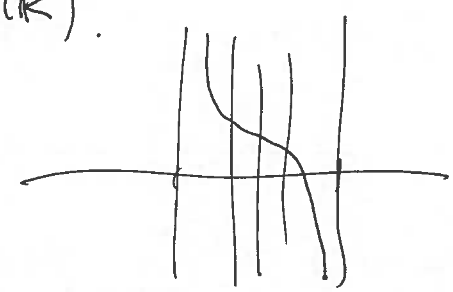
[Nadler]: Arboreal singularities w/ goal of computing $Sh_{\Lambda}(M)$.

[Shende-Treumann-Zaslow, Ng, Sivek, H. Williams] studied $Sh_{\Lambda}(\mathbb{R}^2) / Loc(\mathbb{R}^2)$ as invt. of Λ, & related + other invts., such as augmentations of LCH.

↑ smooth Leg. knot $\subset \mathbb{R}^3$, std.

II. Construction of $Brane_L$: a locally constant sheaf of categories on exact Lag'n L w/ fiber $Mod(\mathbb{K})$.

Rmk: $NZ : Fuk(T^*M) \to Sh(M)$ relying on Floer theory:

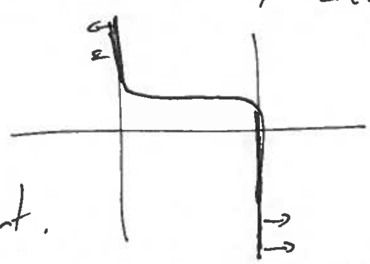


T^*M

define sheaf corr. L as, roughly, having stalks $CF^*(L, T^*_x M)$

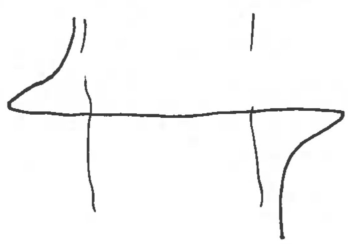
Start from a closed exact Lag'n $L \subset T^*M$. w/ L eventually conic.

meaning $pdq = \alpha|_L = df$; s.t. & $f|_{\text{conic end}} = \text{constant}$.



(L could be compact).

After a $\overset{\text{large}}{\text{}}$ Reeb perturbation: $f \to -\infty$ near the ends of L.



Legendrian lifting

$\tau = -1$ $(q, p; t, \tau)$

$$L \overset{i}{\hookrightarrow} T^*M \times \mathbb{R} = J^1(M) \subset T^*(M \times \mathbb{R}).$$

(q,p) t w/ $\alpha = -dt + pdq$ (now, Cone(iL) is a conic Lag'n in $T^*(M \times \mathbb{R})$.

$\Downarrow$

$(q,p) \longmapsto ((q,p); f(q,p))$.

& note it's Legendrian: $\alpha|_{i(L)} = -df + pdq|_L = 0$.

Let's denote the Legendrian by $\mathbb{L}$; &

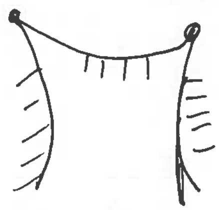
goal: $\mathrm{Sh}_{\mathbb{L}}(M\times\mathbb{R}) \big/ \mathrm{Loc}(M\times\mathbb{R}) \underset{\text{"NZ"}}{\overset{\pi_*}{\longrightarrow}} \mathrm{Sh}(M)$

where $\pi : M\times\mathbb{R} \to M$

$\underline{\mathrm{Brane}}_L$: Do a front projection:

$$\mathbb{L} \longrightarrow M\times\mathbb{R}$$
$$(q,p,t) \longmapsto (q,t)$$

ex:

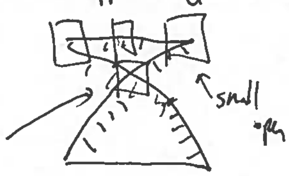


front: has some wave front singularities.
(assume finite to 1): then $\mathbb{L}$ can be recovered as negative conormals to front.

(recall: $\mathbb{L} \subset T^*(M\times\mathbb{R})$ $\tau = 1$.
neg. conormal means $\xi(\partial_t) = -1$)

One tool: Kashiwara–Schapira's theory, but difficult to generalize to ∞-cat. setting.

Different approach: U: $\pi^{-1}(U)\cap\mathbb{L} = \cup\,\Omega_i$.



small open

$\mathrm{Brane}_L : (\mathrm{Open}(L))^{op} \longrightarrow \mathrm{St}_{\mathbb{K}}$ cat. over $\mathbb{K}$.

(presheaf, needs to satisfy a sheaf condition)

 two separate parts.

By local constancy, it's enough to know values on small contractible opens, & verify that small restrictions give equivalences

So, for Ω a small contractible open ~~set~~ in $L/\mathbb{L}$, what's $\quad U \subset M\times\mathbb{R}$ small ball around $\pi(\Omega)$

$\pi^{-1}(U) \subset T^*M\times\mathbb{R}$

$$\mathrm{Brane}_L(\Omega) = \mathrm{Sh}_\Omega(U) \big/ \mathrm{Loc}(U)$$

$\Omega \subset \hat{S}^*T^*U$

independent of choice of small contractible U.

(depends on front being a finite to 1 projection
& locally front is injective on $\mathbb{L}$).

(cf Kashiwara–Schapira)
(proven using contact transformations to straighten out

~~$\Rightarrow \mathrm{Sh}_\Omega(\ )/\mathrm{Loc}(\ ) \overset{\text{total}}{\simeq} \mathrm{Sh}_\Omega(\ )/\mathrm{Loc}(-)$~~

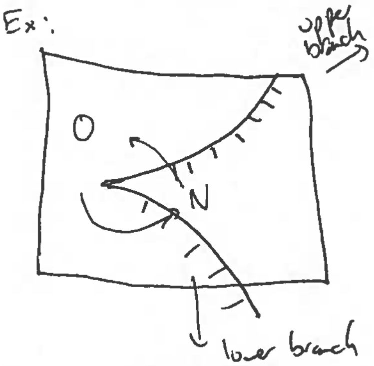
Lemma: (1) $\mathrm{Brane}_L(\Omega) \simeq \mathrm{Mod}(\mathbb{K})$ but, non-canonically

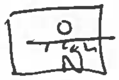
(2) $\Omega' \subset \Omega$ small contractible opens

$\Rightarrow \mathrm{Brane}_L(\Omega) \xrightarrow{\sim} \mathrm{Brane}_L(\Omega')$

$\Rightarrow \mathrm{Brane}_L(\Omega)$ defines a locally constant sheaf w/ fiber $\mathrm{Mod}(\mathbb{K})$

Ex:



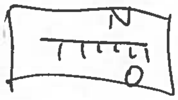


$$\longrightarrow \hookleftarrow Sh_{\square}(U)/Loc(U)$$

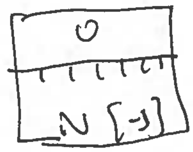
$$Sh^{q}_{\square}(U)/Loc(U)$$

$$\cap$$

$$Sh_{\square}(U)$$



$\longleftrightarrow$



shift of degree related to Maslov index.

Thm 1: [J.-Treumann]:

$$\mathrm{Brane}_L : L \longrightarrow B\mathrm{Pic}(\mathbb{K})$$

(view as classifying map for Brane_L sheaf w/ fiber $\mathrm{Mod}(\mathbb{K})$)

universally factors as

$$\begin{tikzcd} L \arrow[r] \arrow[d] & B\mathrm{Pic}(\mathbb{K}) \\ B\mathrm{Pic}(\mathbb{S}) \arrow[ur] & \end{tikzcd}$$

$\leftarrow \mathbb{S} \to \mathbb{K}$ b/c $\mathbb{S}$ initial in ring spectra

Thm: Brane_L factors thru:

$$L \longrightarrow U/O \xrightarrow[BJ]{} B\mathrm{Pic}(\mathbb{S})$$

stabilized Gauss map

$T^*M \times T^*\mathbb{R} \times T^*\mathbb{R} \times \cdots$

$L \times \mathbb{R} \times \mathbb{R} \times \cdots$

BJ: delooping of J hom.

& $J: \mathbb{Z} \times BO \longrightarrow \mathbb{Z} \times BGL_1(\mathbb{S})$

takes a stable vec. bdle to sphere bdle by Thom construction

Rmks: 1) Only requires L immersed

2) works for any Leg. Λ, not just lift of exact L.

Thm: (1) There is a functor, called microlocal monodromy:

$$\mu mon : Sh_{\Lambda}(M \times \mathbb{R}) / Loc(M \times \mathbb{R}) \xrightarrow{\sim} \Gamma(L, \mathrm{Brane}_L)$$

(2) If Brane_L is trivial, then after choosing a trivialization, ~~get~~ there's a functor
(fully faithful)

$$NZ: \mathrm{Loc}(L) \longrightarrow Sh_{\Lambda_{``L^\infty"}}(M)$$

[~~Gait~~ ~~Gait~~ Guillermou: case L cpct exact Lag'n].

Rmk: trivializing $\mathrm{Brane}(L)$ means identifying $\mathrm{Brane}_L \simeq \mathcal{L}oc_L$, ↙ the trivial $\mathrm{Mod}(\mathbb{k})$ sheaf on L. (sheafy version of $\mathrm{Loc}(-)$ ↑ subset of L.)

$$\begin{array}{c} \Gamma(L, \mathrm{Brane}_L) \simeq \mathrm{Loc}(L) \\ \downarrow \mu_{mon}^{-1} \\ Sh_{\mathbb{L}}(M \times \mathbb{R}) // \mathrm{Loc}(M \times \mathbb{R}) \xrightarrow{\pi_*} Sh_{\Lambda}(M), \end{array}$$

so $(1) \Rightarrow (2)$.

Usual Brane structure is ~~the usual structure~~ a special case of this, via noting that:

$$B\mathrm{Pic}(\mathbb{Z}) = \cancel{B\mathrm{Pic}}\ B\mathbb{Z} \times B^2 \mathbb{Z}/2.$$