T. Pantev, Descent and equivalences in non-commutative geometry — JAS, 3/16/2017

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Goal: Concoct a descent formalism for sheaves without the use of generators
(useful in several ways in the mirror symmetry program, for instance FM equivalences between spaces w/o generators...)
(Rmk: usual descent statement is e.g., Barr-Beck thm for $X = \bigcup U_\alpha$; implicitly uses generators $U_\alpha = \operatorname{Spec} A_\alpha$ ...)

nc-geometry: Bondal's philosophy: A nc space/$\mathbb{C}$ is a $\mathbb{C}$-linear dg category with all colimits.

Notation: (non-standard): $D(X)$ - category
X - nc space (though actual space may not be there in general)

Typical examples: a) X scheme/$\mathbb{C}$ $\rightsquigarrow$ $D(X) :=$ dg enhancement of $D_{qcoh}(X)$

b) A - dg (A_∞) algebra/$\mathbb{C}$, then $D(X) = \operatorname{Spec}_{nc}(A \ldots)$
$:= A\text{-mod}^{sf}$ ← semi-free A-modules.

Thm (Bondal-van den Bergh): If $X/\mathbb{C}$ separated scheme of finite type, $\forall\ Z \subset_{closed} X$, the full subcat. $D_Z(X) \subset D(X)$ is an affine nc space (meaning $\cong$ $A\text{-mod}^{sf}$, some A).
↑ sheaves w/ support on Z

Rmk: Problem: doesn't work for X not separated or not fin. type. (want a replacement of this)

Want to extend this to some algebraic understanding of $D(X)$ or $D_Z(X)$ when X is not quasi-separated, ...

Main tool: (Kontsevich-Rosenberg) nc spaces are often described by nc localizations

Def: A morphism of dg algebras $\phi : A \to B$ is called an nc-localization if the canon. map $B \otimes_A^{(\mathbb{L})} B \longrightarrow B$ is a quasi-isomorphism in $B\text{-mod-}B^{op}$. (of B-bimodules)

Example: If $X/\mathbb{C}$ - separated fin. type scheme,

$Z \overset{i}{\subset} X$ Zariski closed,

$U = X - Z \overset{j}{\hookrightarrow} X.$

Then, we have a recollement diagram:

$$\begin{tikzcd} D(U) \arrow[r, shift left=3ex, leftarrow, "j^*"] \arrow[r, "j_*" description] \arrow[r, shift right=3ex, leftarrow, "j_!"'] & D(X) \arrow[r, shift left=3ex, "i^*"] \arrow[r, leftarrow, "i_*" description] \arrow[r, shift right=3ex, "i^!"'] & D_Z(X) \end{tikzcd}$$

If A - dg algebra computing $D(X)$, then $j^*A \in D(U)$ w/ $\mathrm{End}(j^*A) =: B$ is a generator for $D(U)$,

and the natural map

$$A \xrightarrow{\varphi} B = \mathrm{Hom}(j^*A, j^*A)^{op} \text{ is an nc localization.}$$

<u>Construction</u>: If $A \to B$ nc localization, then get a natural subcategory.

$$C_{A,B} \subset A\text{-mod}$$
$$\| $$
$$\{M \in A\text{-mod} \mid M \otimes_A B = 0\}$$

has all colimits, but in general has no compact objects. The only general object is the cone from $A \to B$, but this need not be compact.

<u>Example</u>: affine line $\mathbb{A}^1 \setminus 0$.

(1) $A = \mathbb{C}[x]$, $B = \mathbb{C}[x, x^{-1}]$

(2) $A = \mathbb{C}[x]$, $B = \mathbb{C}\left[x, \left\{\frac{1}{x - x_i}\right\}_{x_i \in S}\right]$ ← essentially <u>arbitrary</u> (not nec. countable) subset of $\mathbb{C}$, may have accumulation, etc.

$A \to B$ is always a nc localization.

Properties:

(1) If $A \to B$ nc localization, $\Rightarrow$ then $A^{op} \to B^{op}$ nc localization.

(2) If $A \to B$, $A' \to B'$ two nc. localizations, $\Rightarrow$ then

$$A \otimes A' \to \mathrm{Cone}(A \otimes B' \oplus A' \otimes B \to B \otimes B')[1]$$

is a nc localization.

(e.g., $U_1 \subset X_1$, $U_2 \subset X_2$, then $(U_1 \times X_2) \cup (X_1 \times U_2) \subset X_1 \times X_2$).

(3) $\underrightarrow{\mathrm{colim}}$ preserving Functors $\mathrm{Fun}(C_{A,B}, C_{A',B'})$

$$\wr\wr$$

$$C_{A^{op} \otimes A',\ \mathrm{Cone}\left(\begin{smallmatrix} A^{op} \otimes B' \\ \oplus \\ B^{op} \otimes A' \end{smallmatrix} \longrightarrow B^{op} \otimes B'\right)[1]}.$$

(4) $HH_\bullet$, $HH^\bullet$ of $C_{A,B}$ make sense. (as cones on $HH_\bullet$) (but shouldn't define it as $HH_\bullet(C_{A,B})$ as usual category).

Rmk: (Def'n of HH° is something like "$\mathrm{Rhom}_{A\text{-}A?}(\mathrm{cone}(A \to B), \mathrm{cone}(A \to B))$

or $\mathrm{Cone}(A \to B) \otimes^{\mathbb{L}}_{A\text{-}A} \mathrm{cone}(A \to B)$. for $HH_\circ$)

$\mathrm{Cone}(A \to \varphi^* B)$

Zariski descent:

Note: X-scheme / $\mathbb{C}$

$\mathcal{E}$ - presheaf on $\underline{X}$; have usual notion of descent, but this requires effectivity of descent data on flat or Zariski covers.

However: If $\mathcal{E}$ - q.coh. presheaf of $\mathcal{O}$-modules, there is a different sheaf condition we can write.

If X- scheme, a good cover is $\{U_\beta\}_{\beta \in P}$ ~~by~~ by Zariski opens such that

$\uparrow$ poset.

- U_β - affine
- $U_\beta \subset U_{\beta'}$, $\beta \le \beta'$

satisfying

(1) $\bigcup_{\beta \in P} U_\beta = X$

(2) $\forall\ \beta_1, \beta_2 \Rightarrow U_{\beta_1} \cap U_{\beta_2} = \bigcup_{\gamma \le \beta_1, \beta_2} U_\gamma$

("locally contractible").

This $\Rightarrow \forall\ x \in X$,

$I_x = \{\beta \in P \mid x \in U_\beta\}$ has a contractible realization $|I_x|$.

A presheaf of $\mathcal{O}_X$-modules on X gives a collection of modules

$\{\mathcal{F}_\beta\}_{\beta \in P}$ ~~satisfying~~ + maps

$\mathcal{O}(U_\beta) \otimes_{\mathcal{O}(U_{\beta'})} \mathcal{F}_{\beta'} \longrightarrow \mathcal{F}_\beta$ ~~[illegible]~~. $\beta \le \beta'$

If $\mathcal{F}$ a q.coh sheaf (or complex thereof), $\Rightarrow$ these are quasi-isos.

Define

$\mathcal{A} = \mathfrak{A}_{\mathcal{C}} =$ category with $\mathrm{ob}\, \overset{\mathcal{A}}{\overset{\shortparallel}{\mathfrak{A}_{\mathcal{C}}}} = P$

\mathfrak{A}

$$\& \ \mathrm{Hom}_{\mathcal{A}}(\beta, \beta') = \begin{cases} \mathcal{O}(U_{\beta'}) & \beta \le \beta' \\ 0 & \beta \not\le \beta' \end{cases};$$

so $\{\mathcal{F}_\beta\} \in \mathcal{A}$-mod, e.g., can define

presheaves

$P.\,Sh(X, \{U_\beta\}) = \mathcal{A}$-mod & sheaves

$Sh(X, \{U_\beta\}) =$ full subcat. of $\mathcal{A}$-mods $\mathcal{F}$ with

$$\mathcal{O}(U_\beta) \underset{\mathcal{O}(U_{\beta'})}{\otimes} \mathcal{F}_{\beta'} \xrightarrow{\sim} \mathcal{F}_\beta .$$

Then, $D(X) \cong Sh(X, \{U_\beta\})$.

Introduce the notation $Sh(X, \{U_\beta\}) := \mathcal{C}_{\mathcal{A}, \mathcal{B}}$. coming from nc localization

want to argue that this is an nc localization situation, e.g.,

We want to show that this comes from a nc localization

$\mathcal{A} \longrightarrow \mathcal{B}$?? what's this? (e.g., restriction $\mathcal{A}$-mod $\to \mathcal{C}_{\mathcal{A}\mathcal{B}}$-mod is sheafification?)

(Ex: would get $\mathcal{A}$-mod from family Floer theory, & would like to produce sheaves from this, just starting from $\mathcal{A}$-mod)

For this we need to look at

$T' : \mathcal{A}$-mod $\to \mathcal{C}_{\mathcal{A},\mathcal{B}} \subset \mathcal{A}$-mod, which is given by

$- \otimes \mathrm{Cone}(\mathcal{A} \to \mathcal{B})[-1]$ (why T'? the monad is usually called T, T' the "dual" adjoint.)

and T' is a coidempotent comonad. (so it should be right adjoint to inclusion of sheaves).

Remark: (or 'key lemma'): If A-dg algebra, & $T' : A\text{-mod} \circlearrowleft$ which is

equipped w/ co-unit ~~of~~ morphism $\varepsilon' : T' \to \mathrm{Id}$

$\Rightarrow$ If, $\forall$ $M \in A$-mod,

$T' \circ T'(M) \to T'(M)$ - q. iso., φ

$\Rightarrow T'$ is a coidempotent comonad (inverting gives the comonadic structure)

(Key result: there is a universal formula for such a T', w/ universal constants)

<u>Example</u>: Gluing X (<u>separated</u>) out of two affine (Zariski) opens

$$X = U_1 \cup U_2$$

w/ $U_{12} = U_1 \cap U_2$ affine.

Have
$$\mathcal{A} = \left\{ \begin{tikzcd} \mathcal{O}_1 \arrow[dr] & & \mathcal{O}_2 \arrow[dl] \\ & \mathcal{O}_{12} & \end{tikzcd} \right\} \qquad \left(\begin{array}{l} \mathcal{O}_{12} \otimes_{\mathcal{O}_1} \mathcal{O}_{12} \xrightarrow{\sim} \mathcal{O}_{12} \\ \\ \mathcal{O}_{12} \otimes_{\mathcal{O}_2} \mathcal{O}_{12} \xrightarrow{\sim} \mathcal{O}_{12} \end{array} \right.$$

this uses Zariski opens in a key way (doesn't work in flat or étale topologies!

Rmk: ∃ different approach, working w/ modules over coalgebras, which should apply to stacks. But, difficult to solve.)

$$\mathcal{A}\text{-mod} = \left\{ \begin{tikzcd} M_1 \arrow[dr, dashed] & & M_2 \arrow[dl, dashed] \\ & M_{12} & \end{tikzcd} \right.$$

$M_i \in \mathcal{O}_i$-modules $\quad M_{12} \in \mathcal{O}_{12}$, & $\dashrightarrow$ is a map after restriction, e.g.,

$$M_1 \dashrightarrow M_{12} \iff \mathcal{O}_{12} \otimes_{\mathcal{O}_1} M_1 \to M_{12}$$

& same for $M_2 \dashrightarrow M_{12}$,

want to construct sheaf condition intrinsically w/o referring to realization.

$$Sh = \left\{ \begin{tikzcd} M_1 \arrow[dr, dashed] & & M_2 \arrow[dl, dashed] \\ & M_{12} & \end{tikzcd} \;\middle|\; \begin{array}{l} \mathcal{O}_{12} \otimes_{\mathcal{O}_1} M_1 \xrightarrow{\sim} M_{12} \\ \\ \mathcal{O}_{12} \otimes_{\mathcal{O}_2} M_2 \xrightarrow{\sim} M_{12} \end{array} \right\}.$$

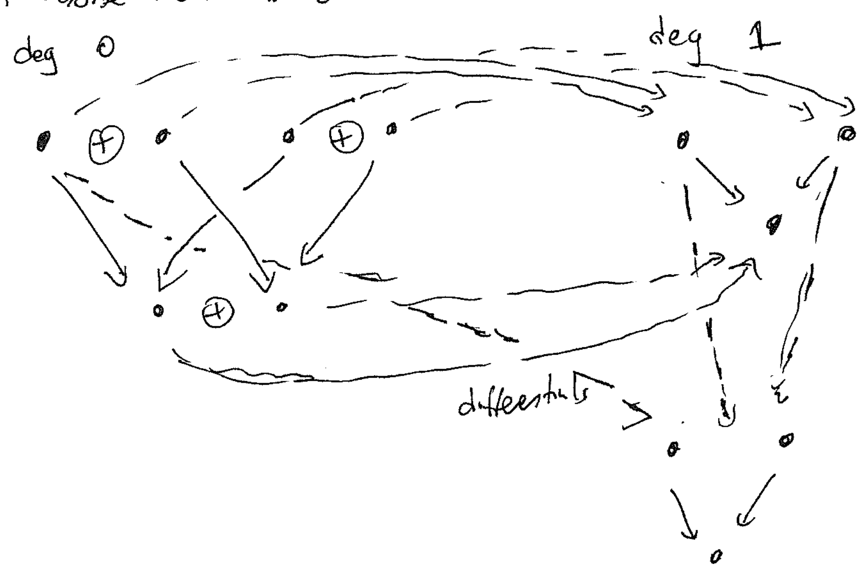
want to construct $T' : \mathcal{A}\text{-mod} \circlearrowleft \mathcal{A}$ which lands in sheaves, & is <u>identity</u> on sheaves. subject of.

In this case, simple formula (in general, very complicated):

If
$$M = \begin{tikzcd} M_1 \arrow[dr, dashed] & & M_2 \arrow[dl, dashed] \\ & M_{12} & \end{tikzcd}, \text{ then}$$

$$T'(M) = \left[\begin{tikzcd} \overset{0}{M_1 \oplus \mathcal{O}_{12} \otimes M_2} \to \overset{1}{M_{12}} \arrow[dr] & & \mathcal{O}_{12} \otimes \overset{0}{M_1} \oplus M_2 \to \overset{1}{\underline{M_1}} \arrow[dl] \\ & \underset{0}{(\mathcal{O}_{12} \otimes M_1 \oplus \mathcal{O}_{12} \otimes M_2)} \to \underset{1}{M_{12}} & \end{tikzcd} \right].$$

Need to describe the counit ε' :



e.g., construct a map

$T(M) \to M$

$\iff$ extension

$M[1] \to ? \to T(M) \ldots$

to describe counit, need to describe extension by $M[1]$

Look at all possible maps to $M[1]$: 6 constants to fix (for morphisms) ~~equations~~. have eq'ns defining what is a counit

write down a system of equations & solve it. (not obvious one can, ~~but one can~~): There is a universal formula.

(works for e.g., "cubical curves", & even more general setting.).

get interesting rational numbers. $\exists$ a unique symmetric (w.r.t. all arrows) $\mathbb{Q}$ sol'n.

are $\mathbb{Z}$, may have to do something asymmetric.