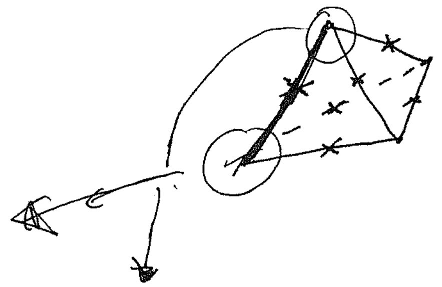
B. Siebert, Topology and real structures in toric degenerations 3/17/2012

w/ Gross, H. Argüz (real str., Kato-Nakayama spaces)

H. Ruddat (periods + tropical cycles),

M. Gross, P. Hacking, S. Keel (K3, theta functions).

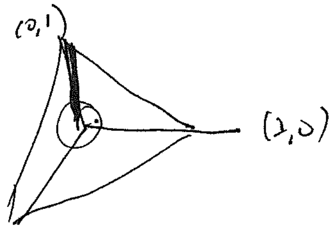
Running example: quartic

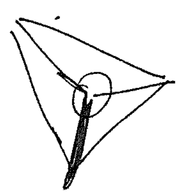


$B, \mathcal{P} = \{\sigma\}$

$\iota : B \setminus \{x_i\} \hookrightarrow B$

charts ($\mathbb{Z}$-affine):





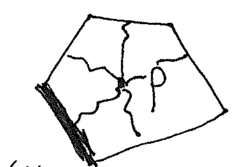
$\rightsquigarrow$ monodromy

$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$

(could split into 4 points the

Λ $\mathbb{Z}$-tangent vectors.

In general,



monodromy $m \longmapsto m + a_{\omega\rho} d_\rho \cdot d_\omega$

$\in N$ (includes 0) $\quad v \quad$ stretch

$\Lambda_\rho^\perp \qquad \Lambda_\omega.$

"a loop around ω through 2-cells adjacent to ρ"

$\updownarrow$

monodromy: one number for each (edge, codim 1 cell)

Nice class of singularities which are indecomposable:

"Simple" (indecomposable) singularities $\Rightarrow$ $a_{\omega\rho} \in \{0,1\}$, —

Here, $a_{\omega\rho} = 4$.

[GS'07]: get $f_{\rho,v} \in \mathbb{C}[\Lambda_\rho] \simeq \mathbb{C}[z_1^{\pm 1}, \dots, z_{n-1}^{\pm 1}]$ & integers

$\kappa_\rho \in \mathbb{N} \setminus \{0\}$

& "gluing data" $s = (s_{\sigma\rho} : \Lambda_\rho \xrightarrow{?} \mathbb{C}^\times)$

$\rightsquigarrow$ canonical toric degeneration $\mathfrak{X}$

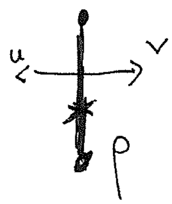
get canonical toric degeneration

$$\mathfrak{X} \supset X_0 = \bigcup TV(\sigma) \quad \text{(polarized family)}$$

$$\downarrow \qquad\qquad \searrow \qquad\qquad \| \qquad + \text{ theta functions}$$

$$S = \operatorname{Spec} \mathbb{C}[[t]] \ni 0 \qquad \underbrace{\bigcup \mathbb{P}^2}_{4} \qquad \vartheta_m \in \Gamma(\mathcal{O}_{\mathfrak{X}}(k)), \quad m \in B(\tfrac{1}{k}\mathbb{Z}).$$

Local equation:



$$\boxed{uv = f_{\rho v} \cdot t^{\kappa_\rho}.}$$

or can generalize κ_ρ to take values in a monoid, then $\mathfrak{X}$ lives over completion of toric.

Example: $s = 1$, $f_{\rho,v} = 1 + w^4$ $\quad \int w$

$$\rightsquigarrow \mathfrak{X} = \text{Dwork pencil} \quad g(t)\cdot(\vartheta_0^4 + \cdots + \vartheta_3^4) + \vartheta_0 \cdots \vartheta_3 = 0$$

fix this fcn. by period integrals.

what's the meaning of $f_{\rho,v}$, κ_ρ, s etc.?

Thm [GS'03]: $\{(f_{\rho,v,s})\}/\text{iso} = H^1(B, \iota_* \check{\Lambda} \otimes \mathbb{C}^*)$.

For $B = S^2$ with (24) simple singularities, we know

$$H^1(\iota_* \check{\Lambda}) = \mathbb{Z}^{20}, \quad H^1(\iota_* \check{\Lambda} \otimes \mathbb{C}^*) = (\mathbb{C}^*)^{20}.$$

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projection:

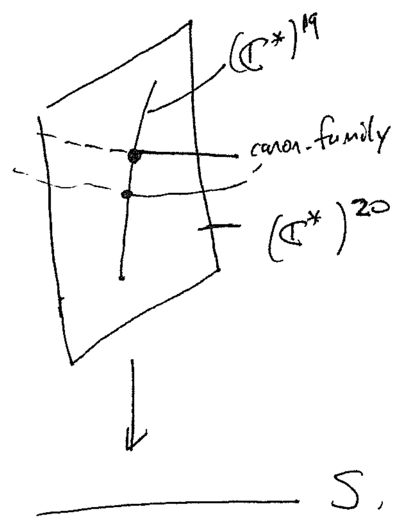
$$ob_{\mathbb{P}} : H^1(\iota_* \check{\Lambda} \otimes \mathbb{C}^*) \to H^2(B, \mathbb{C}^*),$$

$$\qquad\qquad\qquad \ni c_B$$

where c_B "radiance obstruction" $\in H^1(B, \iota_* \Lambda)$

$$\| $$

$$[0 \to \mathbb{Z} \to \mathrm{aff}_{\mathbb{Z}} \to \iota_* \check{\Lambda} \to 0]$$



$\mathbb{R}$-structures: the whole construction plays well with subrings (never introduce denominators or anything).

$$\mathfrak{X}/\mathbb{R} \Longleftrightarrow s \in H^1(\iota_* \check{\Lambda} \otimes \mathbb{R}^\times) \subset H^1(\iota_* \check{\Lambda} \otimes \mathbb{C}^*) \qquad [\text{A.-S. '16}].$$

$\uparrow$ in a way compat. w/ std. $X_0/\mathbb{R}$

"(possibly singular) Momentum map":

$$\begin{array}{ccc} \mu: X_0 & \longrightarrow & B \\ \cup & & \cup \\ TV(\sigma) = X_\sigma & \longrightarrow & \sigma \end{array}$$

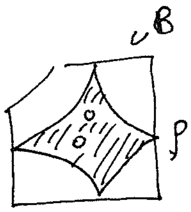
(non-standard for $s \neq 1$).

(can use this to build 2-cycles on X_0 transverse to 1-skeleton).

In X_0, have log singular locus $Z = \bigcup (f_{\rho,v} = 0) \xrightarrow{\mu} \mathcal{A} \subset B$ amoeba image.

$Z \subset (X_0)_{\text{sing}}$

$n=3$



$n=2$

Dwork:

other s: (zero locus (may) splits into different absolute values).

Fact: Affine structure on $B \setminus \mathcal{A}$:

$\updownarrow$

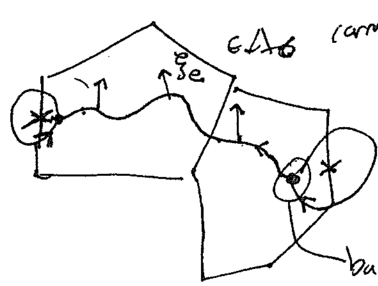
$$\operatorname{Arg} f_{\rho,v} : (\mathbb{C}^*)^{n-1} \setminus Z \longrightarrow U(1)$$

$$\text{''} \quad \operatorname{Hom}(\Lambda_\rho, \mathbb{C}^*)$$

has to do with "the order function on Amoeba complements".

Tropical cycles [Ruddat-S.]:

$$H_1(B \setminus \mathcal{A}, \Lambda) \ni \beta_{\text{trop}}$$



$\mathcal{A}$ = 1:x points } $n=2$ picture

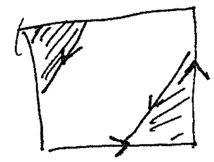
$\rightsquigarrow \beta_0 \in \underset{n}{\overset{}{H_2}}(X_0, \mathbb{Z})$ by looking at

$\xi_e^t \in \mathrm{Hom}(\Lambda_\sigma, U(1))$

$\cong$

$\bigcup U(1)^{n-1}$

$\mathrm{div}(\xi_e)$



$\rightsquigarrow (S^1)^{n-1}$ bundle over edge e of β_{trop}. Near vertices $\rightsquigarrow$ glue in 2-chains

~~(these two cycles are homologous to glade.~~

Using retraction maps

$X_t \longrightarrow X_0$ $\quad \beta_0 \rightsquigarrow (\beta_t)|_{t=0}$ well-defined up to $\mathbb{Z}\cdot f$, where

$t \neq 0$

f = SYZ-fibre

= μ-fibre.

Thm [RS 14]:

canonical n-form on $\mathfrak{X}$

$$(\star) \quad \boxed{\exp\left(2\pi i \int_{\beta_t} \Omega\right) = \langle s, \beta_{trop} \rangle \cdot t^{\langle \delta, \beta_{trop} \rangle}}$$

how β_{trop} wraps under monodromy around ~~t~~ t-circle

gluing data

$$H^1(\iota_* \Lambda^\vee \otimes \mathbb{C}^*) \otimes H_1(\iota_* \Lambda) \longrightarrow \mathbb{C}^*.$$

$\delta \in H^1(B, \iota_* \Lambda^\vee)$: monodromy action

$\beta_t \longmapsto \beta_t + \langle \delta, \beta_{trop} \rangle$

κ_ρ

(where $uv = f_\rho \cdot t^{\kappa_\rho}$)

"how many Dehn twists to do when one goes around."

these are the only periods that compute the canonical coordinates.

(to get more periods $\longleftrightarrow$ curve counts, need higher dimensional cycles)

Application: $\mathrm{Pic}(X_t)$

K3: $H_1(B, \iota_* \Lambda) \simeq U^{\oplus 2} \oplus (-E_8)^{\oplus 2} \simeq \mathbb{Z}^{20}$

Symington $\longrightarrow$ (explicit basis, essentially using what we've now calling "tropical cycles")

& further: $H_2(K3, \mathbb{Z}) \cong \langle s, f \rangle \oplus U^{\oplus 2} \oplus (-E_8)^{\oplus 2}$

s = SYZ-section

From ~~Thm (A)~~, if $s = 1$

$$\Rightarrow S^{+} \subset H_1(i_* \Lambda) \text{ give integral } (1,1) \text{ classes}$$

shows this gives

$$\Rightarrow \mathrm{Pic}(X_t) \simeq \mathbb{Z}^{19} \quad (\longrightarrow \text{DNV-family})$$

(distinguished algebro-geometric families).

~~(B)~~. Next, want to understand

$$(\underline{X_t})_{\mathbb{R}} \subset \underline{X_t}$$

$(f_{p,v}, s)$

$\updownarrow$

Kato–Nakayama space (or Betti realizations) only depends on log structure $(X_0, \mathcal{M}_{X_0})$

$$X_0^{KN} \underset{\text{as sets}}{=} \mathrm{Hom}\left(\Pi^{T}, (X_0, \mathcal{M}_{X_0})\right) \quad \text{topological space}$$

$\uparrow$ pointed space

polar log point $(pt, \mathbb{R}_{\geq 0} \times \mathbb{C}^* \longrightarrow \mathbb{C})$

$(r, e^{i\varphi}) \longmapsto re^{i\varphi}$.

$$\begin{array}{c} X_0^{KN} \\ \downarrow {\scriptstyle \text{compose w/ map to point}} \\ X_0 \\ \downarrow \mu \\ B \end{array}$$

Fact: [Nakayama–Ogus '10 + toric local models [GS'07]].

Say $\begin{array}{c}\mathfrak{X} \\ \downarrow \\ D\end{array}$ disk analytic family. Restrict! $\begin{array}{ccc}\mathfrak{X}_{S^1} & \subset & \mathfrak{X} \\ \downarrow & & \downarrow \\ S^1 & \subset & D\end{array}$ disk

Then

$$\begin{array}{ccc} \mathfrak{X}_{S^1} & \underset{\text{homeo.}}{\cong} & X_0^{KN} \\ \downarrow & & \downarrow \\ S^1 & = U(1) \cong & KN\text{-space of } 0 \in S. \end{array}$$

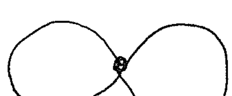
(CY case) (In general case, may have to contract some real blowups of boundaries to get the homeo?)

$U(1) \ni \theta$ fixed phase gives $X_0^{KN}(\theta)$.

θ tells you how to twist.

$\cancel{X}^{KN}(\theta)$

Example:



two $\mathbb{P}^2$'s.

~~(cont'd)~~

$\downarrow \mu$

$B =$

$TV(6)^{KN}$

Thm [AS'16]:

a) Over $B \setminus \mathcal{A}$, X_0^{kN} is homeomorphic (as top. torus bundles) to $\mathrm{Hom}(\mathcal{P}, U(1))$

for $[0 \to \mathbb{Z} \to \mathcal{P} \to \underset{\downarrow}{i_* \Lambda}] = \delta \in \mathrm{Ext}^1(i_*\Lambda, \mathbb{Z}) = H^1(i_*\check{\Lambda})$
$\qquad\qquad\qquad\qquad 0$

$\theta = $ const, get $\mathrm{Hom}(\Lambda, U(1)) \subset \mathrm{Hom}(\mathcal{P}, U(1))$.

(but non-canonically)

(b) Over $\{\text{vertices}\} \cup \bigcup \mathrm{Int}\, \sigma = B'$:

$$\left(X_0^{kN}\big|_{B'} \to B'\right) \underset{\text{canon.}}{=\!=} \mathrm{Hom}^0(\hat{\mathcal{P}}, U(1)),$$

for $[0 \to \mathbb{Z} \oplus U(1) \to \hat{\mathcal{P}} \to i_*\Lambda \to 0] = (\delta, \mathrm{Arg}(s)) \in \mathrm{Ext}^1(i_*\Lambda, \mathbb{Z} \oplus U(1))$.

(c) For $s \in H^1(i_*\check{\Lambda} \otimes \mathbb{R}^*)$, then

$$(X_0^{kN})_{\mathbb{R}} \subset X_0^{kN} \text{ as } \mathrm{Hom}(\hat{\mathcal{P}}, \{\pm 1\}^n).$$

K3's: $B = S^2$, 24 points. Dichotomy: — either $\mathrm{Arg}\, s = 1 \Rightarrow$ (i) $(X_0^{kN}) \overset{\theta=1}{=} (\Sigma_0 \cup \Sigma_1) \xrightarrow{4:1} S^2$

genus 10

1:1

(Σ_0 section)

otherwise (ii) connected

$b \sum_i b_i(\Sigma_1) = 22 \doteq b_2(K3)$!

c.f. [Cast-Bern & Matessi paper(s)].

explanation in terms of Leray ss's ...