J. Solomon, Open descendent integrals

- Motivation
- genus 0
- higher genus

i.e. for target space = pt.
"ψ integrals"

"ODI"

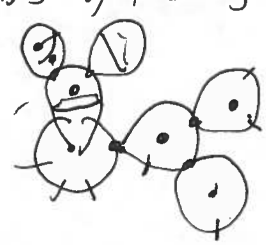
Motivation • '14: Pandharipande - S. - Tessler Analog of Witten's conjectures for open descendent integrals
(open KdV hierarchies, open Virasoro)
- construct in genus 0, proof of conj. in genus 0.

- S-Tessler (in progress) construct higher genus ODI.
- Tessler '15: analog of Kontsevich's combinatorial model for ODI.
- ~~Tessler~~ Buryak-Tessler '15: combinatorial model $\Rightarrow$ open Witten conjectures.
- Zernik '17: - compute structure coefficients of Fukaya A_∞ algebra of $\mathbb{RP}^{2n} \subset \mathbb{CP}^{2n}$ in terms of genus 0 ODI.
  - Welschinger's real enumerative invariants + higher dim'l generalization in style of S-Tukachinsky.
  - ? compute higher genus Fukaya category / open GW invts
- Giterman-S. (in progress): Deform Fukaya category using descendant classes
  $\rightsquigarrow$ descendant Open GW invts, TRR
  ? open Virasoro constraints (for target $\neq$ pt.)

Genus zero: Have $\mathcal{M}_{0,k,l}$ = stable disks Σ w/ k boundary marked pts $x_1, \dots, x_k$, and l interior marked points $z_1, \dots, z_l$

cpt. smooth manifold w/ corners.

typical element:



/ reparam.

$\dim \mathcal{M}_{0,k,l} = k + 2l - 3$

$\dim PSL_2(\mathbb{R}) = \mathrm{Aut}(D^2)$

$\mathbb{L}_i \longrightarrow \mathcal{M}_{0,k,l}$ tautological line bundle

$\mathbb{L}_i \big|_{(\Sigma, x, z)} = T^*_{z_i} \Sigma$.

(Remark: If $\mathbb{L}$ is a complexification of a real line bundle, $c_1(\mathbb{L})$ is torsion, so can't get $\mathbb{Q}$ numbers. This is why we don't consider $\mathbb{L}$ over bd. marked pts. $x_1, \dots, x_k$ (tautological lines))

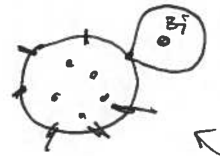
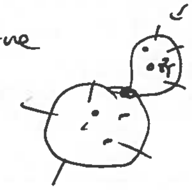
Want $(*) = \int_{\mathcal{M}_{0,k,l}} c_1(\mathbb{L}_1)^{a_1} \cdots c_1(\mathbb{L}_l)^{a_l}$; (to be non-zero, need $(**)\ 2\sum a_i = \dim \mathcal{M}_{0,k,l}$)

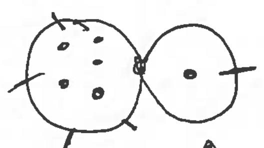
not defined because $c_1(\mathbb{L}_i) \in H^*(\mathcal{M}_{0,k,l})$, but we need $c_1(\mathbb{L}_i) \in H^*(\mathcal{M}_{0,k,l}, \partial\mathcal{M}_{0,k,l})$ to integrate..

$\overset{\text{''}}{\text{deg. integrand}}$

lift?

Not clear how to make the lift.

$\partial\mathcal{M}_{0,k,l}$ ~~$=$~~ $\supsetneq$ $\Big\{$  $\Big\}$ on this bit of boundary, $\mathbb{L}_i$ is the cplxification of a real bundle, so would be good. But: can also have 

E.g., $\Big\{$  $\Big\}$

1-D stratum, would need to say how to trivialize $\mathbb{L}_i$ as pass through bd. and forth.

Write $E = \bigoplus_{i=1}^{l} \mathbb{L}_i^{\otimes a_i}$. Then, $\mathrm{Euler}(E) = \prod_{i=1}^{l} c_1(\mathbb{L}_i)^{a_i}$.

$$\& \quad (*) = \int_{\mathcal{M}_{0,k,l}} \mathrm{Euler}(E).$$

So, suffices to provide $\mathrm{Euler}(E) \in H^d(\mathcal{M}_{0,k,l}) \longleftarrow H^d(\mathcal{M}_{0,k,l}, \partial\mathcal{M}_{0,k,l})$.

Idea: Choose a nowhere vanishing section ~~$s \in \Gamma(E|_{\partial\mathcal{M}_{0,k,l}})$~~ $s \in \Gamma(E|_{\partial\mathcal{M}_{0,k,l}})$. (always exists b/c $\mathrm{rk}\, E = \dim \mathcal{M} \geq \dim \partial\mathcal{M}$!)

Define $\mathrm{Euler}(E,s) = PD(\tilde{s}^{-1}(0))$ where $\tilde{s} \in \Gamma(E)$ extends s.
$\in H^d(\mathcal{M}, \partial\mathcal{M})$

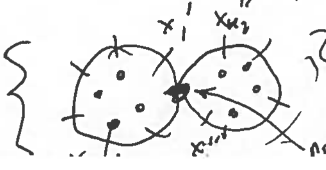
$\mathrm{Euler}(E,s)$ ~~depends~~ doesn't change if change s by nowhere vanishing homotopy.

Still, a priori many choices. (want to make a canonical one)

A component of $B \subset \partial\mathcal{M}_{0,k,l}$ is diffeomorphic to $B \simeq \mathcal{M}_{0,k_1,l_1} \times \mathcal{M}_{0,k_2+1,l_2}$

$k_1 + k_2 = k$

$l_1 + l_2 = l$.

$\Big\{$  $\Big\}$

Observation:

$$\underset{\operatorname{rank}(E)}{\underset{\shortparallel}{2\sum a_i}} = 2l + \overset{\dim \mathcal{M}_{0,k,l}}{\overset{\downarrow}{k}} - 3$$

$\Rightarrow$ k odd. (for such an integral to be non-vanishing)

$\Rightarrow$ WLOG, k_1 odd, k_2 even.

In particular, k_1 odd $\Rightarrow k_1 \neq 0 \Rightarrow$ can forget node without destabilizing a component containing z_i

$$\begin{array}{c} E \searrow \mathcal{M}_{0,k_1+1,l_1} \times \mathcal{M}_{0,k_2+1,l_2} \\ \pi \big\downarrow \text{ forget one side of node} \\ \hat{E} = \bigoplus_{i=1}^{k} \mathbb{L}_i^{\otimes a_i} \to \mathcal{M}_{0,k_1,l_1} \times \mathcal{M}_{0,k_2+1,l_2} \end{array}$$

$\Rightarrow E \cong \pi^* \hat{E}$. (since forgetting doesn't destabilize)

Def: $s \in \Gamma(E|_{\partial \mathcal{M}_{0,k,l}})$ is canonical if ~~$s = \pi^*\hat{s}$~~ for each boundary component $B \subset \partial \mathcal{M}_{0,k,l}$ $\exists\, \hat{s} \in \Gamma(\hat{E})$ s.t. $s = \pi^* \hat{s}$.

Thm (P-S-T): $\exists$ canonical (multi)-section, unique up to nowhere vanishing homotopy.

b/c rank of $\hat{E} = \dim \mathcal{M}_{0,k_1,l_1} \times \mathcal{M}_{0,k_2+1,l_2} + 2$.

Example: $$\int_{\mathcal{M}_{0,3,1}} \mathrm{Euler}(\mathbb{L}, s) = -1 + 3 = 2.$$

Note: $\dim \mathcal{M}_{0,3,1} = 3 + 2\cdot 1 - 3 = 2$

$\mathcal{M}_{0,3,1}$

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