T. Yu, The Frobenius conjecture in dimension 2 (joint w/ S. Keel) 3/16/2017

Plan: §1. The Frobenius construction of GHK

§2. Main results

§3. Structure constants via non-archimedean geometry

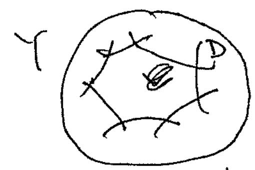
§4. Finiteness

§5. Compactification & extension of the mirror family

§1. Goal: Simple conjectural construction of mirror to an affine log CY variety w/ maximal boundary.

Y: conn. smooth proj. var./$\mathbb{C}$., & $D \in |-K_Y|$ effective s.n.c. divisor containing at least 1 0-stratum "maximal boundary"

ex



$U := Y \setminus D$ affine; call it a log CY w/ max. ∂.

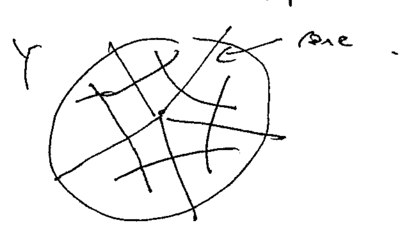
Let $R := \mathbb{Z}[NE(Y)]$.

Want: mirror family $\mathcal{V} / \operatorname{Spec} R$.

Idea: $\mathcal{V} = \operatorname{Spec} A$, A: free R-module with basis in $B(\mathbb{Z})$ where $B :=$ fan of (Y, D), cone of dual graph over D!

integer points ($\longleftrightarrow$ "divisorial valuations" (a priori, such a valuation may not have center an irred. component of D))

can write $A = \bigoplus_{p \in B(\mathbb{Z})} R \cdot \theta_p$.



Q: how to obtain an R-alg. structure on A?

GHK's observation: for each $n \geq 2$, $\exists$ a natural multilinear map

$\langle \cdot, \dots, \cdot \rangle : A^n \longrightarrow R$ given by counting rational curves:

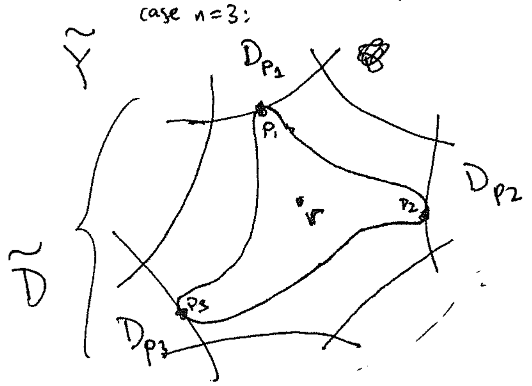
Given $p_1, \dots, p_n \in B(\mathbb{Z})$, $\beta \in NE(Y)$, $\exists$ toric blowup $\pi : (\tilde{Y}, \tilde{D}) \longrightarrow (Y, D)$

such that p_i has divisorial center $\underset{\text{irred. component}}{\tilde{D}_{p_i}} \subset \tilde{D}$

Let $c(p_1, \dots, p_n, \beta) := \# \{ (\mathbb{P}^1, (p_1, \dots, p_n, r)) \xrightarrow{f} \tilde{Y}$ s.t.

- $f^*(D_{p_i}) = m_i p_i$ (& doesn't intersect other divisors), multiplicity, defined by $P_i = m_i P_i^{\text{prim}}$.
- $f_*[\mathbb{P}^1] = \beta$.
- domain fixed generic modulus
- $f(r) =$ a fixed generic point $y \in \tilde{Y}$ $\}$

case $n=3$:



Prop: This set is finite. # is well-defined. (doesn't e.g., depend on choice of blow-up, modulus, ...)

Therefore, can define the multilinear map $\langle , \dots , \rangle_n : A^n \longrightarrow R$

$$(\theta_{p_1}, \dots, \theta_{p_n}) \longmapsto \sum_{\beta \in NE(Y)} c(p_1, \dots, p_n, \beta)\, z^\beta .$$

Conjecture (GHK):

(1) The pairing $\langle , \dots , \rangle_n$ is non-degenerate.

(2) $\exists !$ comm. R-alg. structure on A s.t.

- $1_A = \theta_0$
- ~~trace~~ $\mathrm{trace}(a_1 \cdots a_n) = \langle a_1, \dots, a_n \rangle_n$

  meaning: coeff. of 1_A in this product

(3) ~~V~~ $\mathrm{Spec}\, A \to \mathrm{Spec}\, R$, restricted to $T_{\mathrm{Pic}(Y)} \subset \mathrm{Spec}\, R$, is a family of affine log CY varieties with maximal ∂.

2: Main result:

Thm 1 (Keel-Y): The conjecture holds in dim. 2.

Idea: Construct the structure constants of A by counting hol. disks, as follows:

Given $P_1, \dots, P_n \in B(\mathbb{Z})$

define
$$\theta_{P_1} \cdots \theta_{P_n} = \sum_{Q \in B(\mathbb{Z})} \Big(\sum_{\gamma \in NE(Y)} \eta(P_1, \dots, P_n, Q, \gamma)\, z^\gamma \Big) \theta_Q \quad (\star)$$

where, given

$\tau : U \to B$ SYZ fibration

$\eta(P_1, \dots, P_n, Q, \gamma)$ counts holom. disks;

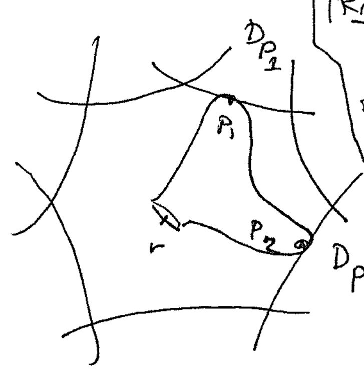
$(\Delta, (p_1, \dots, p_n, r)) \longrightarrow \tilde{Y}$ s.t.

- $r \in \partial\Delta$, $p_1, \dots, p_n \in \Delta^\circ$
- $f^*(D_{P_i}) = m_i p_i$ as before
- $\tau f(\partial\Delta) =$ a fixed point b near to ray $\overrightarrow{OQ}$ (so $\partial\Delta$ lives in a particular torus fiber)
- $f_*[\Delta/\partial\Delta] = \gamma$, $f_*[\partial\Delta] = Q$ (can think of $Q \in H_1(\text{torus})$) gen. stalk is $\cong (\mathbb{C}^*)^n$, & identify $B(\mathbb{Z})$ w/ $H_1(T^n)$. toric at ∞
- generic domain, & $f(r)$ a fixed generic point inside $\tau^{-1}(b)$ (the fiber torus).

(Auroux):

$\boxed{\text{Rmk}}$ If didn't fix domain, might encounter a regularity issue; need to count stable disks w/ component in divisors; for fixed modulus, this does not arise.

$n = 2$ case:



Difficulty: The set depends a priori on various choices.

Strategy: To make it precise, we use the theory of non-archimedean enumerative geometry developed in [Y.]'s thesis.

Step 1: Replace the SYZ fibration $\tau: U \to B$ by the NA SYZ fibration. (non-archimedean & analytic) (which is canonical & explicit)

Construction: $k = \mathbb{C}((t))$, $U_k = U \otimes_{\mathbb{C}} k$, U_k^{an} NA space $/k$.

For a component $D_i \subset D$, locally D_i is defined by a function u_i.

$\mathrm{val}\, u_i$ gives a continuous function on U_k^{an}, & the

NA SYZ fibration $\tau: U_k^{an} \longrightarrow B$ canonical.

$x \longmapsto (\mathrm{val}\, u_i(x))$

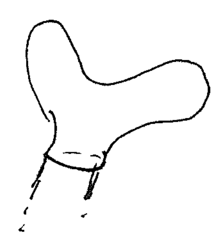
Step 2: We want to count holom. disks in Y_k^{an}

$(\Delta, (p_1, \dots, p_n, r)) \xrightarrow{f} \tilde{Y}_k^{an}$ such that

- $p_1, \dots, p_n \in \Delta^\circ$
- $f^*(D_{p_i}) = m_i p_i$
- $\tau f(\partial\Delta)$ = a fixed point $b \in B$ near the ray $\overrightarrow{OQ}$ ("$f(\partial\Delta)$ lies in a torus fiber")
- $f_*[\Delta/\partial\Delta] = \gamma$, $\tau f(\text{nhood of } \partial\Delta) = b \nearrow Q$.
- generic domain, $f(r)$ a fixed generic point.

Trouble: This set is infinite.

Solution from [Y.]'s thesis: impose a regularity condition on the boundary; specifically, we ask: by analytic continuation at ∂, our disk extends all straight (meaning its image in B is straight w.r.t. $\mathbb{Z}$-aff. structure on B.)



Thm 2: The space of hol. disks in $\tilde{Y}_k^{an}$ satisfying $\{\vdots$ + boundary regularity condition is a finite set.

Rmk: This finite set decomposes into subsets determined by 'spine' of the disks.

Trouble 2: (cont'd).

Trouble 2: "Extending straight" on the left of $\overrightarrow{OQ}$ differs from "extending straight" on the right side of $\overrightarrow{OQ}$.

Solution: Define yet another regularity condition: by analytic continuation, want our hol. disk to extend straight w.r.t. a toric model $\pi: Y \to \bar{Y}$

$\uparrow$ blowdown of Y to a toric variety $\bar{Y}$.

Then,

Thm 3: Counts using "left regularity condition"

= - - - "toric regularity condition"

= - - - "right regularity condition."

Cor: The counts $\eta(P_1, \ldots, P_n, Q, \gamma)$ are well defined.

Thm 4: (Associativity): $(\theta_{P_1} \cdots)(\theta_{P_n}) = \theta_{P_1} \cdots \theta_{P_n}$

add arbitrary parentheses — multiplication w/o parentheses.

§4. Finiteness theorems:

Q: Are the Σ's in ($\star$) finite? (if not, only get a formal algebra instead of an actual algebra).

Thm 5: (Finiteness I): Given $P_1, \ldots, P_n \in B(\mathbb{Z})$, $\exists$ at most finitely many (Q, γ) s.t. $\eta(P_1, \ldots, P_n, Q, \gamma) \neq 0$

Cor: A is an associative commutative R-algebra.

Thm 6: (Finiteness II): A is a finitely generated R-algebra.

§5. $$\begin{array}{c} \operatorname{Spec} A \\ \downarrow \\ \operatorname{Spec} R \end{array}$$ mirror

To compactify, fix F ample divisor on Y s.t. supp $F = D$,

$\rightsquigarrow$ filtration on A given by $A_{\leq s} = \bigoplus_{\{q \in B(\mathbb{Z}) \mid F^{trop}(q) \leq s\}} R \cdot \theta_q$

$\uparrow$ tropical fcn; real-valued fcn on B. — F divisor.

$\rightsquigarrow$ subalgebra $\tilde{A} \subset A[T]$, gen. by aT^s, $a \in A_{\leq s}$ as a submodule.

$\rightsquigarrow$ compactification $\mathcal{X} := \operatorname{Proj}(\tilde{A})$ $\downarrow$ $\operatorname{Spec} R$ is a fibrewise compactification of $\operatorname{Spec} A \downarrow \operatorname{Spec} R$

Now, let

$$R = \mathbb{Z}[NE(Y)], \quad NE(Y)_{\mathbb{R}}^{\vee} = \mathrm{Nef}(Y) \subset N^1(Y,\mathbb{R}), \quad \mathrm{Spec}\, R = TV(\mathrm{Nef}(Y))$$

Nef cone of Y

toric variety assoc. to Nef cone of Y.

Now study larger base:

$$TV(\text{Mori Fan}(Y)) \supset TV(\mathrm{Nef}(Y))$$

in words, this is a fan in N^1 w/ more cones than $\mathrm{Nef}(Y)$

$\{$cones of Mori Fan$\}$ $\longleftrightarrow$ things like $\pi: Y \to Y'$ birational map w/ Y' normal; get cone in N^1 consisting of divisors which are ~~Nef div~~ pulled back from Nef divisors ~~of~~ of Y' + exc. divisors of π.

<u>Thm</u> 7: The compactified family $\begin{matrix}\mathcal{X}\\ \downarrow \\ \mathrm{Spec}\, R\end{matrix}$ extends over $TV(\text{Mori Fan}(Y))$.

<u>Final thm</u>: (Keel-Y): The family $\begin{matrix}\mathcal{X}\\ \downarrow \\ TV(\text{Mori Fan}(Y))\end{matrix}$ has nice properties:

- some kind of smoothness
- when restrict to open torus on base, even better smoothness (restr. to open base is a log CY surface, w.r.t. $\mathcal{D}$, & at worst canonical singularities).

[Non-degeneracy of product follows from Thm 7_Y, by appealing to knowledge of birational things ~].

(Rmk: in higher dimensions $\not\exists$ always a toric model).