Miami '09: Abouzaid

HMS for T^4

(0) $Fuk(T^2) \longleftrightarrow D^b(coh(E))$ Polishchuk–Zaslow

$\omega + iB \longleftrightarrow$ modular param. Polishchuk

(1) B real n-torus — Symplectic torus

Fukaya $TB/\mathbb{Z}^n$. — $T^*B/\mathbb{Z}^n$

HF^* over $\mathbb{C}$ — cplx. mfld.

$\{$coh. sheaves$\} \longleftrightarrow \{$Aff lin. Lagr'ns.$\}$

$\leftrightarrow Ext^*$

for $\pitchfork$ Lagr'ns,

check A_∞ structure

agrees.

(2) Kontsevich – Soibelman

Consider $\{$(unbranched) Lagr'n multisections$\} \longleftrightarrow \{$vector bundles$\}$ over Novikov ring.

$\Rightarrow D^b Coh(Y) \longleftrightarrow D^\pi Fuk.$

For T^4, we (A. – Smith) prove this is an equivalence.

Consider the Tate elliptic curve over $\Lambda_{\mathbb{R}} = \{\sum c_i t^{\lambda_i}\}$

Analytification $\Lambda_{\mathbb{R}}^* / \langle t \rangle$. $\quad \lambda_i \to +\infty$, $c_i \in \mathbb{C}$.
$\lambda \in \mathbb{R}$

$$Fuk(T^2) \cong D^b Coh(E)$$

__Thm:__ (A-Smith) $D^\pi Fuk(T^2 \times T^2) \cong D^b Coh(E^2)$.

__Define__ $Fuk^{naive}(X) = \left\{ \begin{array}{l} \text{Lagranges } L, \text{ maslov } 0, \text{ spin}, \\ \exists J \text{ s.t. } L \text{ does not bound a } J\text{ holo. disc} \end{array} \right\}$

__Thm:__ $D^\pi Fuk^{naive}((T^2)^n) \cong D^b Coh(E^n)$.

(For $n=2$, we're lucky, in that $Fuk^{naive} = Fuk$).

(relying on K-S & maybe Fukaya, can now make same statement for any abelian surface)

__Thm__ (Generation Criterion):

Let $\mathcal{A} \subset Fuk^{naive}(X)$ such that

① $$\begin{tikzcd} QH^*(X) \arrow[r] \arrow[dr, "\cong", "\text{iso}"'] & HH^*(Fuk^{naive}(X)) \arrow[d] \\ & HH^*(\mathcal{A}) \end{tikzcd}$$

smoothness

② The diagonal of $\mathcal{A}$ can be resolved (as a bimodule) by tensor products $\mathcal{Y}_l(K_-) \otimes \mathcal{Y}_r(K_+)$ $(K_\pm \in \mathcal{A})$

$\mathcal{Y}_l(K_-)(L) = CF^*(K_-, L)$, $\mathcal{Y}_r(K_+)(L) = CF^*(L, K_+)$.

$\Longrightarrow \mathcal{A}$ generates $D\mathrm{Fuk}^{naive}(X)$.

(note; if ② is true up to idempotent, Maydan's "smoothness", then $\mathcal{A}$ split-generates $D^{\pi}\mathrm{Fuk}^{naive}(X)$)

(this theorem tells us we don't have to pass to the product, i.e. T^8, to obtain a resolution of the diagonal).

X is CY here, I believe...

(M'au–Wehrheim–Woodward, in progress):

$\mathrm{Fuk}^{\#}(X)$:

objects: $L_{01}, L_{12}, \ldots, L_{k-1,k}$

$$\mathrm{pt.} \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_{k-1} \longrightarrow X_k = X$$

$L_{i,i+1} \subset X_i^- \times X_{i+1}$ reverse ω

in the "naive Fukaya cat.".

Morphisms: $\vec{L} = (L_{01}, L_{12})$, $\vec{K} = (K_{01}, K_{12})$

$L_{01} \subset X_1$, $L_{12} \subset \overline{X_1} \times X$, $K_{01} \subset Y_1$, $K_{12} \subset \overline{Y_1} \times X$.

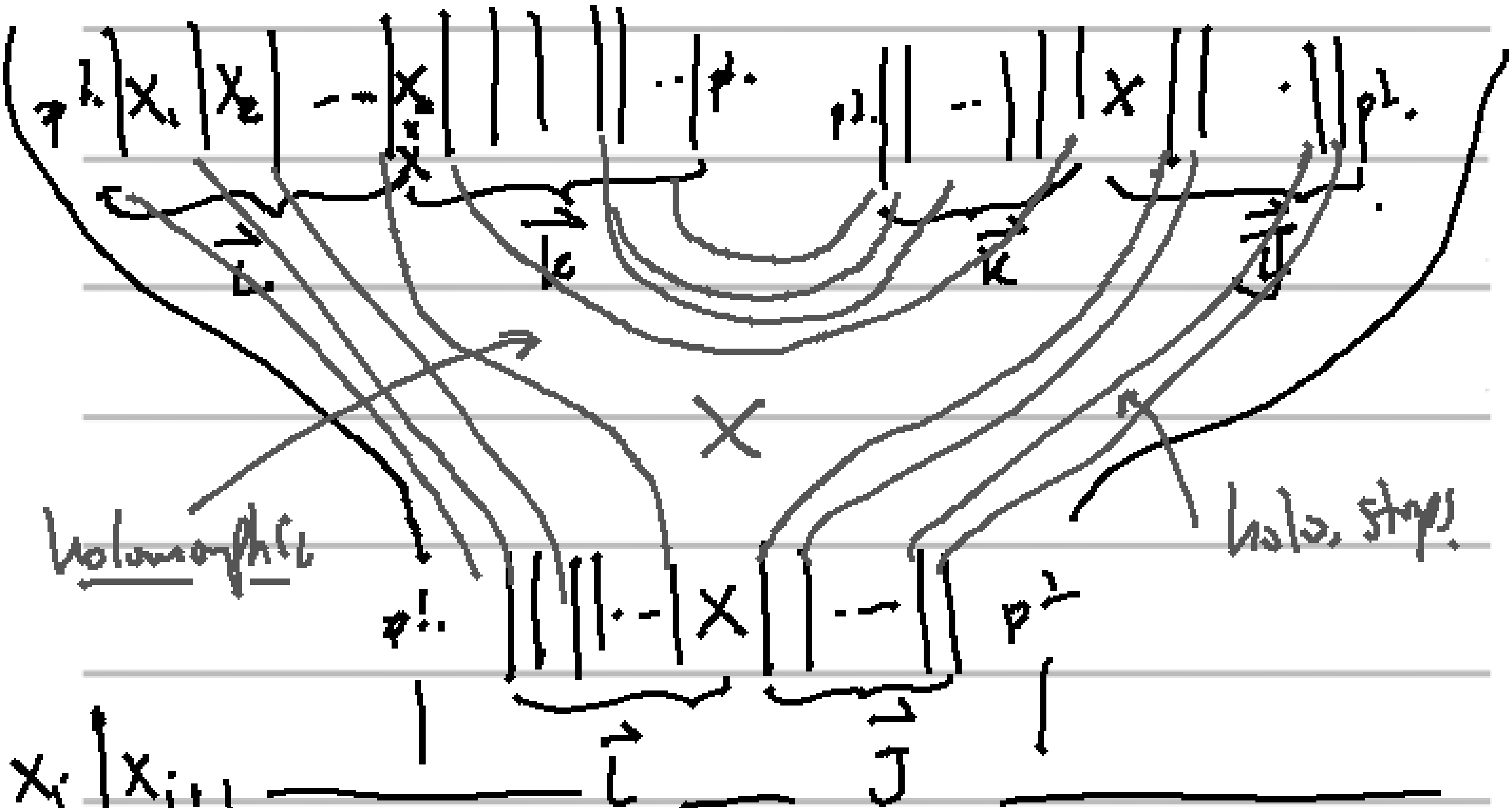
$CF^*(\vec{L}, \vec{K}) =$ pt. $\overset{L_{01}}{|}$ X_1 $\overset{L_{12}}{|}$ X $\overset{K_{12}}{|}$ Y_1 $\overset{K_{01}}{|}$ pt.

gen. by isections $\begin{cases} L_{01} \times K_{12} \subset X_1 \times \overline{X} \times Y_1 \\ L_{12} \times K_{01} \subset X_1 \times \overline{X} \times Y_1 \end{cases}$

<u>Composition</u>: $CF^*(\vec{L},\vec{K}) \otimes CF^*(\vec{K},\vec{J})$

$$\downarrow$$

$$CF^*(\vec{L},\vec{J})$$



$\exists$ functor $Fuk^{naive}(X \times \bar{X}) \longrightarrow End(Fuk^\#(X))$

$$L \longmapsto \left(\{pt\} \to X_0 \to \cdots X_{k-1} \xrightarrow{L_{k,k+1}} X\}\right)$$

$$\downarrow$$

$$\{pt. \to \cdots \to X_{k-1} \to X \xrightarrow{L} X\}$$

ev(strip, --)
$\uparrow$
$L_{i,i+1}$
$\uparrow$
$X_i \times X_{i+1}$

(Can't do something naive b/c no nice formula of tensor product of A_∞ structures as an A_∞-struct.)

Consider $\mathcal{A}^\# \subset Fuk^\#(X)$: - all $X_i = X$
- all $L_{i,i+1} = K_- \times K_+$ or Δ (diagonal).
$K_\pm \in Ob(\mathcal{A})$.

$$pt. \xrightarrow{K_-^1 \times K_+^1} X \to X \to \cdots \xrightarrow{K_-^k \times K_+^k} X.$$

Lemma:
$$\begin{tikzcd} \mathcal{A} \arrow[r, hook] \arrow[d, hook] & \mathcal{A}^{\#} \arrow[d, hook] \\ \mathrm{Tw}\,\mathcal{A} \arrow[r, hook, "\text{equivalence}"] & \mathrm{Tw}\,\mathcal{A}^{\#} \end{tikzcd}$$

$\mathcal{A}(X \times \bar{X}) \rightsquigarrow$ objects: Δ, $K_- \times K_+$ $(K_\pm \in \mathcal{A})$

$\cap$

$\mathrm{Fuk}^{\text{naive}}(X \times \bar{X})$

$\Rightarrow \boxed{\mathcal{A}(X \times \bar{X}) \xhookrightarrow{\text{MWW}} \mathrm{End}(\mathrm{Tw}(\mathcal{A}))}$

Note that $\begin{cases} K_- \times K_+ \longmapsto (L \longrightarrow CF^*(L, K_-) \otimes K_+). \\ \Delta \longmapsto \mathrm{id} \end{cases}$?

Claim: Under the assumption of Thm, MWW is a fully faithful embedding

For $(K_\pm, \Delta)$ ← obvious.

$(K_\pm, K_\pm')$

Reasons: $(\Delta, K_\pm)$ follows from) by Poincaré duality

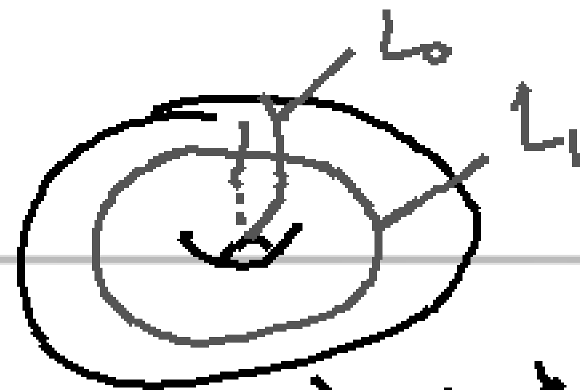
and (Δ, Δ) ⟷ $\boxed{\text{by assumption } QH^* \xrightarrow{\sim} HH^*(\mathcal{A})}$

Smoothness property for $\mathcal{A}$ ⇒ resolution of the diagonal in $X \times \bar{X}$

$$\Delta \in \mathrm{Fuk}(X \times \bar{X}) \xrightarrow{\text{MWW}} \mathrm{End}(\mathrm{Fuk}^{\text{naive}}(X)).$$

$$\overset{\cup}{\mathcal{A}(X \times \bar{X})} \xleftrightarrow{\text{MWW}} \mathrm{End}(\mathrm{Tw}(\mathcal{A})).$$

MS for T^4:

$\mathcal{A}(T^2) =$ 

By P-Z, $Tw^\pi(\mathcal{A}(T^2)) \cong D^b Coh(E)$

$\mathcal{A}(T^4) = \{$ category with objects $L_i \times L_j \}$. By above,

$$\mathcal{A}(T^4) \to End(Tw(\mathcal{A}(T^2))) \cong End(D^b Coh(E)) \hookrightarrow D^b Coh(E^2)$$

image of $\mathcal{A}(T^4)$ split generates $D^b Coh(E^2)$

Can easily check generation conditions.