Miami — Auroux I — SLag fibrations in MS

1. Overview of SYZ
   - → CY case } no instanton corrections
   - → Fano case }
2. Instanton corrections — examples
   Mirror symmetry for pairs
3. Mirrors of blowups
   (... Abouzaid – Ketzarkov, in progress)

SYZ conjecture: $X, X^\vee$ mirror pair carry dual SLag torus fibrations

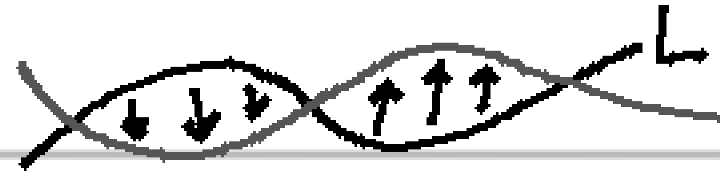
caveats:
- ☆ existence not clear —— large complex str. limit.
- ☆ instanton corrections.

Def: (X^n, J, ω) Kähler mfold is (almost) Calabi-Yau if $K_X \cong \mathcal{O}_X \Rightarrow \exists\, \Omega \in \Omega^{n,0}(X)$ hol. vol. form

(don't require $|\Omega|_g = \text{const.}$)

$L^n \subset X$ is <u>special</u> Lag. if $\omega|_L = 0$, $\operatorname{Im}\Omega|_L = 0$.

<u>Deformations of SLag</u>:



$\omega|_L = 0$

$\operatorname{Im}\Omega_L = 0$

$v \in C^\infty(NL)$

$L_v \omega = 0, \quad L_v \operatorname{Im}\Omega = 0$

$\alpha = -\iota_v \omega \in \Omega^1(L, \mathbb{R})$

$\beta = \iota_v \operatorname{Im}\Omega = \psi \cdot *_g \alpha \in \Omega^{n-1}(L, \mathbb{R})$

$\quad \searrow \psi = |\Omega|_g.$

So, SLag deforms $= \{ v \in C^\infty(NL) \mid -\iota_v \omega \in \mathcal{H}^1_\psi(L) \}$, where

$\mathcal{H}^1_\psi(L) = \{\alpha \in \Omega^1(L,\mathbb{R}) \mid d\alpha = 0,\ d^*(\psi\alpha) = 0\}$

$\cong H^1(L,\mathbb{R}) \cong H^{n-1}(L,\mathbb{R})$

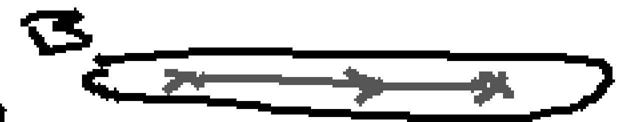
$[\psi * \alpha]$

McLean, Joyce — moduli space of SLags is smooth.

For $L \cong T^n$, might get a local fibration

$$\begin{array}{ccc} T^n & \longrightarrow & X \\ & & \downarrow \pi \\ & & B \end{array}$$

B has 2 affine structures



$T_L B \underset{\text{can.}}{\cong} H^1(L,\mathbb{R}) \underset{\text{can}}{\cong} H^{n-1}(L,\mathbb{R})$

"Sympl" affine coords = sympl areas swept by $H_1(L)$

"C" affine coords = $\int \operatorname{Im}\Omega$ on ——→—— $H_{n-1}(L)$

<u>Dual fibration</u>: $L^\vee = \mathrm{Hom}(\pi_1(L), U(1)) =$

$\{$flat $U(1)$ conn's on $\mathbb{C} \to L\}$

Given SLag fibration $L \to X \xrightarrow{\pi} B$,

dual $M = \{(L,\nabla) \mid L$ SLag fiber of π, ∇ $U(1)$ local system$\}$

$T_{(L,\nabla)} M = \{(v,\alpha) \in C^\infty(NL) \oplus \Omega^1(L,\mathbb{R}) \mid$

$-\iota_v\omega + i\alpha \in \mathcal{H}^1_\psi(L) \otimes \mathbb{C}\}$

$(M, J^\vee)$ integrable $\mathbb{C}$ structure

$\Omega^\vee((v_1,\alpha_1), \dots, (v_n,\alpha_n)) = \int_L (-\iota_{v_1}\omega + i\alpha_1) \wedge \cdots \wedge (-\iota_{v_n}\omega + i\alpha_n)$

holom. $(n,0)$-form

$\omega^\vee((v_1,\alpha_1),(v_2,\alpha_2)) = \frac{1}{\int_L \Omega} \int_L \alpha_2 \wedge \iota_{v_1} \operatorname{Im}\Omega - \alpha_1 \wedge \iota_{v_2} \operatorname{Im}\Omega$

Kähler form

$$\begin{tikzcd} X \arrow[dr, "\pi"'] & M \arrow[d, "\pi^\vee"] & (L,\nabla) \arrow[dl] \\ & B & \end{tikzcd}$$

SLag T^n fibrations

2 affine str's on B interchanged

purely imaginary b/c we don't want to discuss "B-fields"

Ex: $T^2 = \mathbb{C}/\mathbb{Z} + i\tau\mathbb{Z}$

$$\Omega = dz, \int_{T^2} \omega = \lambda$$

S^1 $\qquad$ T^2

$B = S^1$

$\mathbb{C}$-aff. str: τ $\qquad$ sympl: λ

M.S. changes $\lambda \longleftrightarrow \tau$

• HMS $\quad D^\pi \mathrm{Fuk}(X) \simeq D^b \mathrm{Coh}(M)$

$$\mathcal{L}_p \longleftrightarrow \mathcal{O}_p, \ p \in M \text{ skyscraper}$$

$$H^*(T^n, \mathbb{C}) \simeq \mathrm{Ext}^*(\mathcal{O}_p, \mathcal{O}_p)$$

From now on, (X, J, ω) Kähler

$\cup$

$D \in |-K_X|$ normal crossings

restr. on singularities

$\Omega = \sigma_D^{-1} \in \Omega^{n,0}(X \setminus D)$ (simple poles on D)

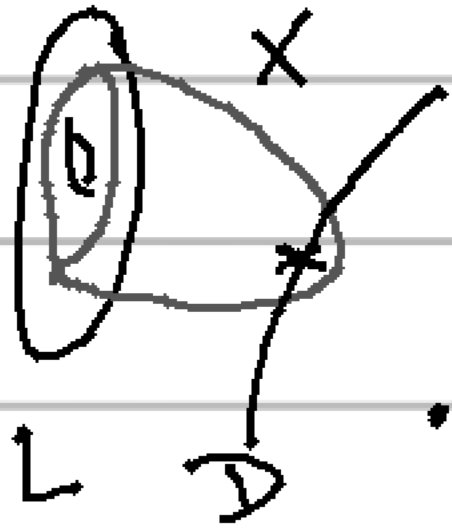
$X \setminus D$ almost CY $\longrightarrow$ mirror M.

Mirror of X should be a Landau–Ginzburg model

$M \xrightarrow[\text{"superpotential"}]{W} \mathbb{C}$.

Construction: $(L, \nabla) \in M \quad (L \subset X \setminus D)$

$\beta \in \pi_2(X, L)$



exp $\dim \mathcal{M}(L,\beta) = n-3+\mu(\beta)$

$\mu(\beta) = 2 \cdot (\beta \cap D)$

Assume for now:

- no (non const.) discs in (X, L) with $\mu(\beta) \leq 0$
- $\mu = 2$ discs regular.

Given $\beta \mid \mu(\beta) = 2$, $\dim \mathcal{M}(L,\beta) = n-1$

$n_\beta(L) = \deg_p ev_*[\overline{\mathcal{M}}_1(L,\beta)]^{vir}$

$= \#$ discs through $p \in L$

$$W(L,\nabla) = \sum_{\substack{\beta \in \pi_2(X,L) \\ \mu(\beta)=2}} n_\beta(L)\, z_\beta(L,\nabla) \in \mathbb{C}$$

$\hat{M}$

$$z_\beta(L,\nabla) = \underbrace{\exp\left(-\int_\beta \omega\right)}_{\mathbb{R}_+} \cdot \underbrace{\mathrm{hol}_\nabla(\partial\beta)}_{U(1)}$$

DANGER $\downarrow$ $\triangle$ Z

$\longrightarrow$ convergence?

$\longrightarrow$ when $\exists$ Maslov 0 discs, W is multivalued / discontinuous.

Rmk: z_β are local holom. coords. on M.

$$d \log z_\beta(\underbrace{v,\alpha}_{T_{(L,\nabla)}M}) = \int_{\partial\beta}(\underbrace{-\iota_v\omega + i\alpha}) \quad \text{(complex. hol. by construction!)}$$

Example: $\mathbb{C}P^2$ or any toric Fano.

$$X = \mathbb{C}P^2,\ D = \{x_0x_1x_2 = 0\},\ \text{toric } \omega.$$

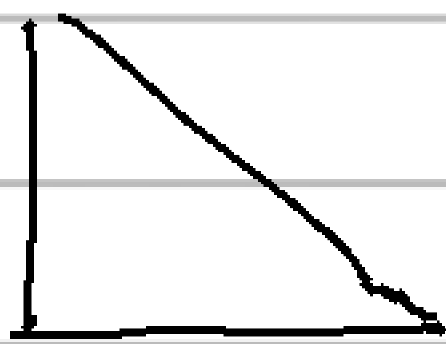
$$X \setminus D \simeq (\mathbb{C}^*)^2,\ \Omega = \frac{dx \wedge dy}{xy}.$$

$$L = S^1(r_1) \times S^1(r_2) \subset (\mathbb{C}^*)^2 \subset \mathbb{C}P^2 \quad \text{SLag}.$$

SLag fibration $T^2 \to X \setminus D = (\mathbb{C}^*)^2$ SLag

$$\downarrow$$

B = orbit space

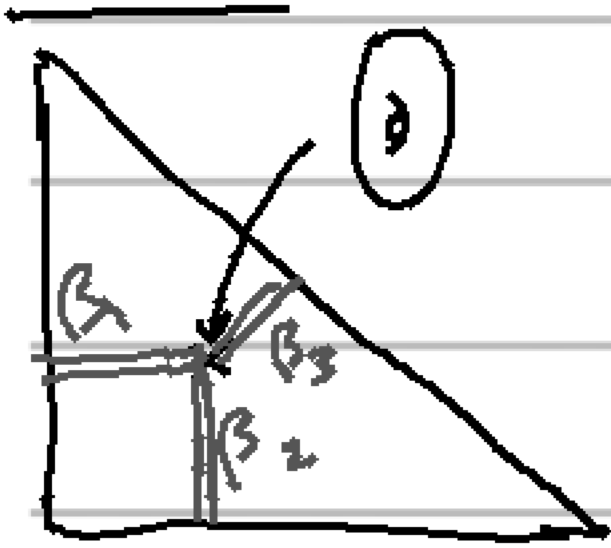


sympl. $B = \mathrm{int}(\Delta)$ ← moment polytope

$\mathbb{C}$ $\quad B = \mathbb{R}^2$, Log map

Dualize

$$M = \{(z_1, z_2) \mid \left(\frac{-1}{2\pi}\log|z_i| \in \mathring{\Delta}\right\} \subset (\mathbb{C}^*)^2$$



$$L = S^1(r_1) \times S^1(r_2)$$

no discs in $(\mathbb{C}^*)^2$ (max. principle)

$\mu = 2$: hit one coord axis

$$D^2(r_1) \times \{pt.\} \subset \mathbb{C}^2 \subset \mathbb{C}P^2$$

$$\{pt.\} \times D^2(r_2)$$

3 families

$n_{\beta_1} = n_{\beta_2} = n_{\beta_3} = 1$. $\quad \sum [\beta_i] = [\mathbb{C}P^1]$

$$z_{\beta_1} = z_1,\ z_{\beta_2} = z_2,\ z_{\beta_3} = \frac{e^{-\text{area}(\mathbb{C}P^1)}}{z_1 z_2}.$$

$(z_\beta = e^{-\int_\beta \omega}\, \mathrm{hol}(\partial\beta))$

So $W = z_1 + z_2 + \dfrac{e^{-\mathrm{Area}(\mathbb{CP}^1)}}{z_1 z_2}$.

This geometric picture / way to construct mirror goes back to [Hori, Cho-Oh, FOOO, ---].

NB: Hori-Vafa have a way of completing the mirror to all of $(\mathbb{C}^*)^n$, using renormalization flow —— mathematically corresponds to enlarging $[\omega_t] = [\omega] + t c_1(X)$.

Fukaya claim: above expansion only works in monotone case, in non-monotone case, you're losing information in the enlargement process about, e.g. which singularities are inside the polytope & which are outside. (e.g. $\mathbb{P}^2$ blown up at a pt.).