Miami '09: Auroux II

<u>Recall</u>:

(X, J, ω) Kähler

D anticanonical, $\Omega \in \Omega^{n,0}(X \setminus D)$

6 $M = \{(L, \nabla) \mid L \subset X \setminus D$ SLAG torus, ∇ flat $u(1)$ local system$\}$.

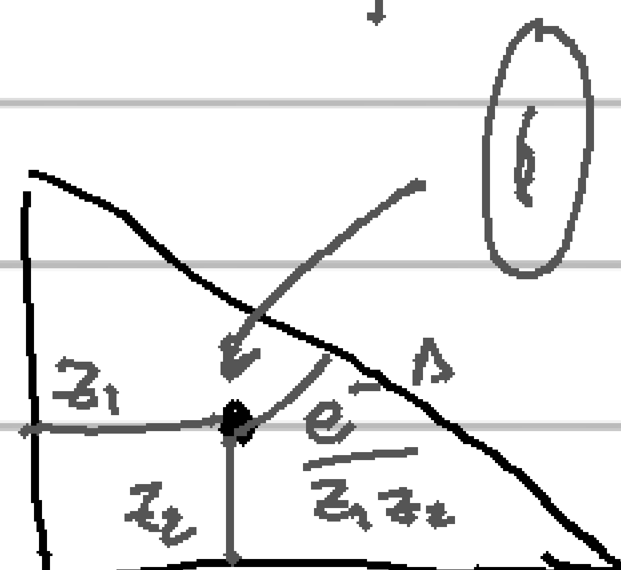
optimistically defined:

$$W(L, \nabla) = \sum_{\substack{\beta \in \pi_2(X, L) \\ \mu(\beta) = 2}} \underbrace{n_\beta(L)}_{\substack{\#\text{ discs} \\ \text{through} \\ p \in L}} \underbrace{z_\beta(L; \nabla)}_{= \exp(-\int_\beta \omega)\, \mathrm{hol}_\nabla(\partial \beta)}$$

<u>Ex</u>: $\mathbb{CP}^2$, $D = \{x_0 x_1 x_2 = 0\}$

$M = (\text{subset})$ of $(\mathbb{C}^*)^2$

$$W = z_1 + z_2 + \frac{e^{-\Lambda = \text{area}(\mathbb{P}^1)}}{z_1 z_2}$$



<u>Interpretation</u>: (W as a way of obstructing mirror).

say $L = S^1(r_1) \times S^1(r_2) \subset (\mathbb{C}^*)^2 \subset \mathbb{CP}^2$

In $(\mathbb{C}^*)^2$: $HF(L, L) \cong H^*(T^2)$

$$(L, \nabla) \longleftrightarrow \mathcal{O}_p, \; p \in M \simeq (\mathbb{C}^*)^2.$$

In $\mathbb{CP}^2$, L is <u>obstructed</u> by holom. discs

<u>Ex</u>:

$\partial x = c \cdot y$, $\partial y = c' \cdot x$, $\partial^2 = z_\beta \cdot \mathrm{Id}$, so can't define $HF(L, L)$ but <u>weakly unobstructed</u>.

On $CF(L,L')$:

FOOO

$$\partial^2(x) = m_0(L')\cdot x - x\cdot m_0(L)$$

$$m_0 = W\cdot 1 \qquad \in CF^*(L,L')$$

this is the weakly unobstructed case.

(So get a bunch of Fukaya categories parametrized by values of W, between different categories, Fuk is obstructed).

<u>Problem</u>:

- $HF((L,\nabla),(L',\nabla'))$ only defined if $W(L,\nabla)=W(L',\nabla')$
- $HF(L,L)$ usually zero.

$$\mathcal{F}(X) \ni (L,\nabla) \longleftrightarrow \mathcal{O}_p, \quad p\in M.$$

$$D^b\cancel{Coh}(M) \rightsquigarrow D^b Sing(M;W) \quad \text{(Orlov)}$$

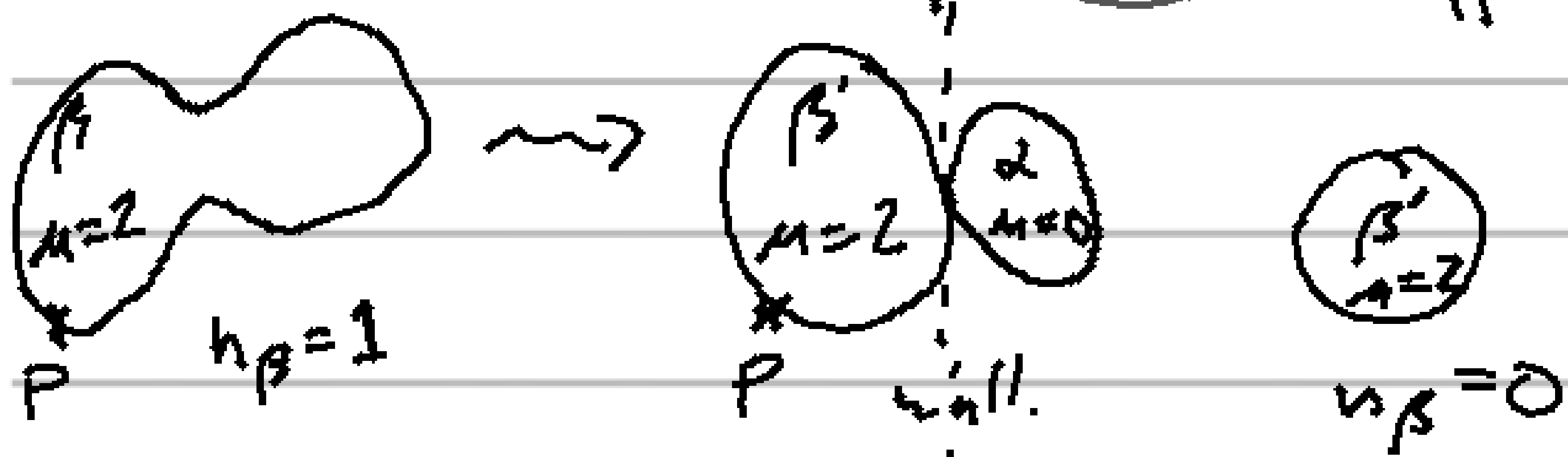
$$= \coprod_{\lambda\in\mathbb{C}} D^b_{Sing}(\{W=\lambda\})$$

$\mathcal{O}_p \neq 0$ only if $p\in \operatorname{Crit} W$.

E.g. $x+y+\frac{c}{xy}$ crit at 3 roots of unity corresponding to Clifford torus & 3 distinguished local systems, the only cases where HF is non-zero.

Wall-crossing (for dummies?)

expected dim $\mathcal{M}(L,\beta) = n-3+\mu(\beta)$ $= 2(\beta \cap D)$



W discontinuous or multivalued (depending on choice of p)

deform L

So $W = $ function of $\begin{cases} L \\ \nabla \\ \text{extra data (e.g. } p\in L) \end{cases}$

Caveat: lot of extra assumptions to get this.

Denis: can't you take $\deg_p(ev_*[\bar{\mathcal{M}}_1(\beta,L)]^{vir})$

<u>Example</u>: $X = \mathbb{CP}^2$, $\Omega = \dfrac{dx\wedge dy}{xy-\varepsilon}$

$D = \{xy=\varepsilon\} \cup \{\text{line at } \infty\}$.

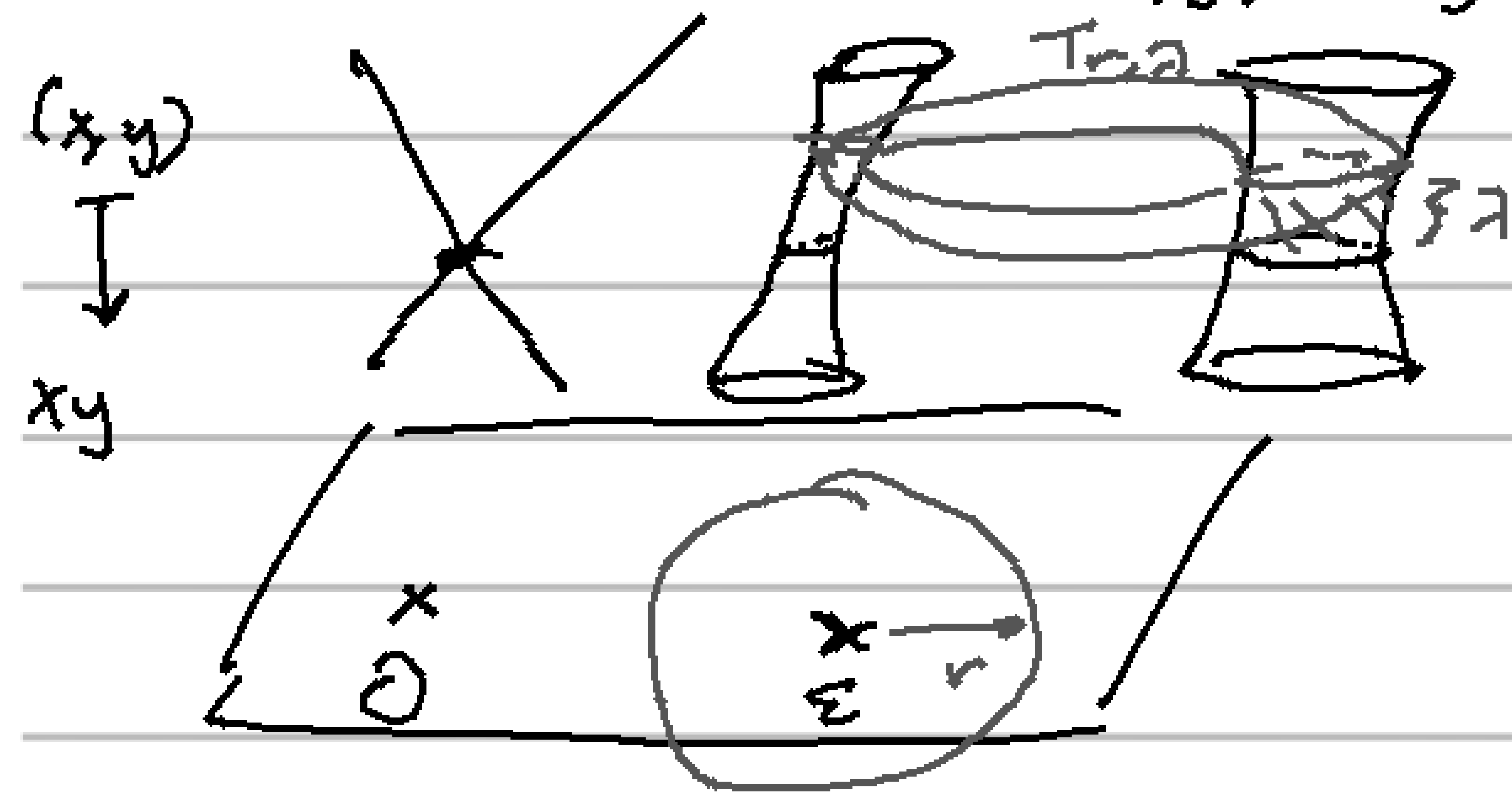
S^1 acts by $e^{i\theta}(x,y) = (e^{i\theta}x, e^{-i\theta}y)$

Moment map $\mu = \dfrac{1}{2}\dfrac{|x|^2-|y|^2}{1+|x|^2+|y|^2}$.

Claim:

$\exists$ S^1-invt. family of SLag tori ("special" not important here).

$$T_{r,\lambda} = \{(x,y)\in\mathbb{C}^2 \,/\, |xy-\varepsilon| = r,\ \mu(x,y)=\lambda\}$$

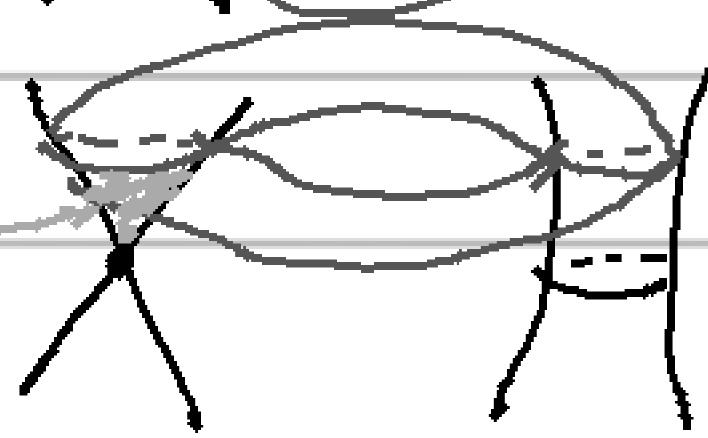


$$T(T_{r,\lambda}) \ni v = (ix, -iy)$$

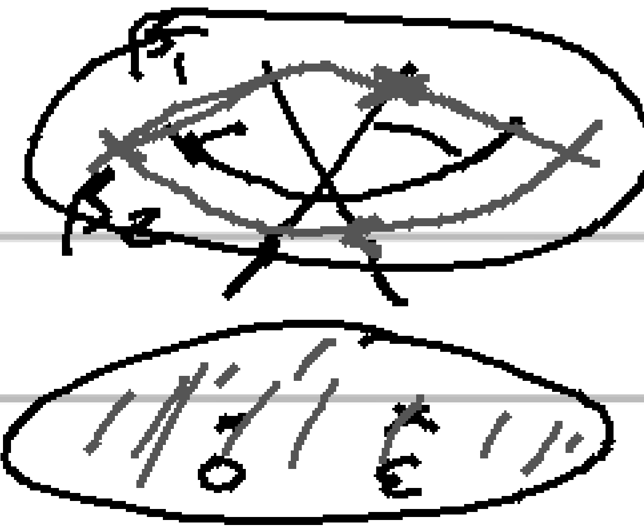
$$\left.\begin{array}{l} \iota_v\omega = d\mu \\ \mathrm{Im}\,\iota_v\Omega = \mathrm{Im}(i\,d\log(xy-\varepsilon)) \end{array}\right\} = 0 \text{ on } T_{r,\lambda}.$$

- $T_{|\varepsilon|,0}$ singular.

- $T_{|\varepsilon|,\lambda}$ bounds a $\mu=0$ disc



For $r > |\varepsilon|$:



$T_{r,\lambda} \sim$ product tori

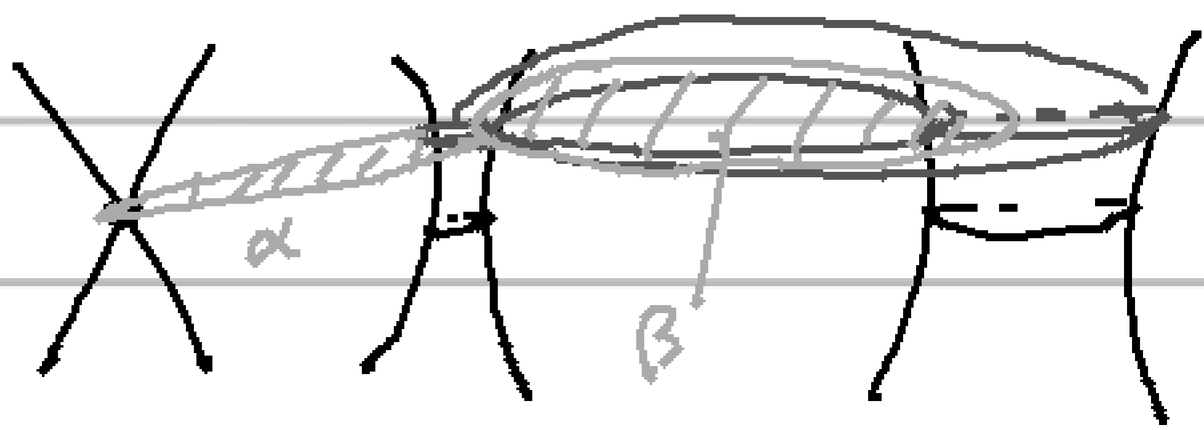
$\Rightarrow$ 3 families of holom. discs as before.

classes $\beta_1, \beta_2, [\mathbb{CP}^1] - \beta_1 - \beta_2$,

$$W = z_1 + z_2 + \frac{e^{-\Lambda}}{z_1 z_2} \quad (*)$$

For $r < |\varepsilon|$, $T_{r,\lambda}$ bounds

4 families (Chekanov torus (can deform to this case?))



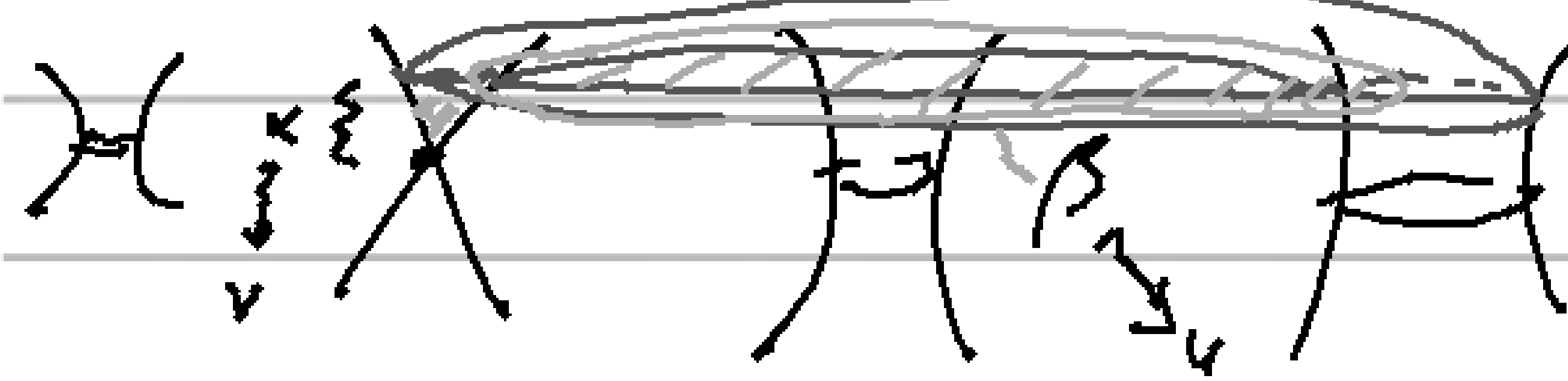
$z_\beta = u$

$z_\alpha = v$.

$n_\beta = 1$, $n_{[\mathbb{CP}^1] - 2\beta \pm \alpha} = 1$, $n_{[\mathbb{CP}^1] - 2\beta} = 2$

$$W = u + \frac{e^{-\Lambda}(1+v)^2}{u^2 v} \quad \text{vs. } (*)$$

As r crosses $|\varepsilon|$, $\lambda > 0$



$\beta \longleftrightarrow \beta_2$ $\qquad$ expect: $u \longleftrightarrow z_2$

$\alpha \longleftrightarrow \beta_1 - \beta_2$ $\qquad$ $v \longleftrightarrow z_1/z_2.$

On $r > |\varepsilon|$ side, create $\beta_1 = \beta_2 + \alpha$

$u \longleftrightarrow z_2(1+v) = z_1 + z_2.$

$v \longleftrightarrow z_1/z_2.$

General Result: (FOOO)

∃ analytic change of variables under which W becomes continuous.

If wall $\longleftrightarrow$ disc in a class α, gluing

$$z_\beta \longleftrightarrow z_\beta \cdot h(z_\alpha)^{[\partial\beta]\cdot[\partial\alpha]}$$

$$h = 1 + O(z_\alpha) \in \mathbb{Q}[[z_\alpha]]$$

(NB: generally multiple covers of $\mu=0$ discs are $\mu=0$ also, have to account for them).

This gluing makes sense as a Kähler mfold, no idea how to put on a Kähler form, yet.

$\lambda < 0$

$u \longleftrightarrow z_1$

$z_1(1+v^{-1}) = z_1 + z_2$

MS for a pair (X,D)

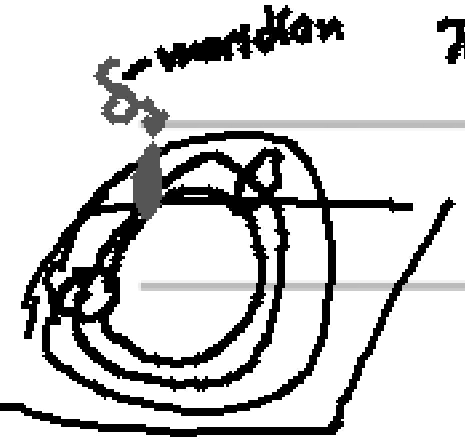
folklore: "fiber of W is mirror to D"

- $D \subset X$ is (almost) Calabi-Yau:
  $|D| \in |-K_X|$ $\quad \omega|_D, \ \Omega_D = \mathrm{Res}_D(\Omega)$

explanation: near D, $\Omega = \Omega_D \wedge d\log \sigma_D + O(1)$

defining section.

Conj: The fibers of the SLag fibration $\pi: X \setminus D \to B$ near ∂B lie in a nbhd of D & are S^1-bundles over SLag tori in (D, ω_D, Ω_D)



D

$$W = z_\delta + o(1).$$

(Assume: X compact, D smooth).

- $\partial M = \{|z_\delta| = 1\}$
  $\big\downarrow \arg(z_\delta)$
  S^1

fiber $M_D = \{z_\delta = 1\} \underset{C^\infty}{\simeq} \{W = 1\}$??

L collapsed on D

∇ pulled back from $\Lambda \subset D$.

$\cong$ SYZ mirror to D

Equivalently: [$\omega \longleftrightarrow \omega + t\, c_1(X)$, enlarges mirror]

$\{W = e^t\}_{t \to \infty}$ family of fibers
approximates SYZ mirrors to
$(D, \omega_D + t c_1(X)_D)$

(N.B. Instanton corrections in X & D not the same!).
To get better statement, need to explain how discs & instanton corrections in X & D are related to each other.

However, symplectically, any smooth fiber of W should be mirror to D (as a cplx. mfold).

Checked (HMS) for
$\mathbb{P}^2$ + blow-ups of $\mathbb{P}^2$ [A-Katzarkov-Orlov].

Abo-Zaid: $(\mathbb{P}^3$, smooth K3) quartic

sometimes antinonical

D has structure of affine mfold w/ singularities. singularities propagate in.

M.S. for K3 only sees walls that come from singularities

this is what tells you the gluing by instanton correcting is compatible!