Miami '09 — Fukaya: Oh-Ohta-Ono

open-closed GW theory

closed GW theory first

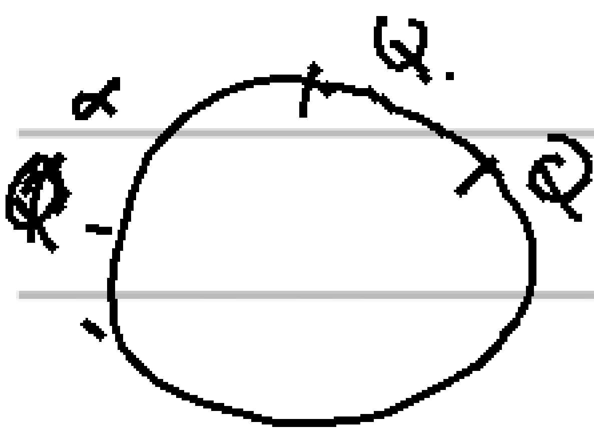
$\Lambda_0 = \{\sum a_i T^{\lambda_i} \mid a_i \in \mathbb{Q}, \lambda_i \geq 0, \uparrow \infty\}$

$\Psi : H(X; \Lambda_0) \longrightarrow \Lambda_0.$

$Q \in H(X; \mathbb{Q})$ $\alpha \in H_2(X; \mathbb{Z})$ $\ell \geq 0$

$n(\alpha, \ell, Q) \in \mathbb{Q}$

$$= \# \left\{ (u, z_1, \ldots, z_\ell) \;\middle|\; \begin{array}{l} u : S^2 \to X \text{ holo.} \\ [u] = \alpha \\ z_i \in S^1,\ u(z_i) \in Q \end{array} \right\} \Big/ \text{aut.}$$



$$\Psi(Q) = \sum_{\alpha, \ell} \frac{T^{\omega \cap \alpha}}{\ell!} n(\alpha, \ell, Q)$$

$Q = \sum w_i f_i$, f_i basis of $H(X; \mathbb{Q})$

$$\langle f_i \cup^Q f_j, f_k \rangle = \frac{\partial^3 \Psi}{\partial w_i \partial w_j \partial w_k}(Q).$$

Now

OC: $L \subset X$ Lag'n submfold

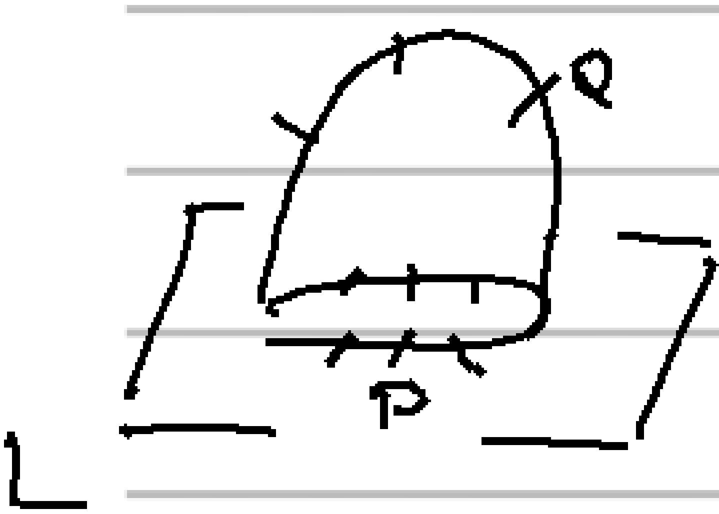
$\Phi : H(X, \Lambda_0) \times H(L, \Lambda_0) \longrightarrow \Lambda_0$

with $Q \in H(X, \Lambda_0)$, $P \in H(L, \Lambda_0)$

$\beta \in H_2(X, L; \mathbb{Z})$, $k \geq 0$, $\ell \geq 0$ gives us

$n(\beta, k, \ell, Q, P) \in \mathbb{Q}$

$$= \# \frac{\left\{ \begin{array}{l} (u, z_1, \ldots, z_\ell, \\ z_1', \ldots, z_k') \end{array} \;\middle|\; \begin{array}{l} u: (D^2, \partial) \to (X, L) \text{ holo.} \\ [u] = \beta \\ z_i \in \mathrm{Int}\, D^2, \; u(z_i) \in Q. \\ z_i' \in \partial D^2 \;\; u(z_i') \in P \end{array} \right\}}{\mathrm{Aut}.}$$



$$\Phi(Q, P) = \sum_{\beta, k, \ell} \frac{T^{\beta \cap \omega}}{k!\,\ell!} \, n(\beta, k, \ell, P, Q).$$

we don't assume respects cyclic ordering. (actually want to do this?)

$$q^{Q,P}_{\mathrm{sym}} : \underset{\substack{\Psi \\ \sum w_i f_i}}{H(X, \Lambda_0)^{\otimes \ell}} \otimes \underset{\substack{\Psi \\ \sum x_i e_i}}{H(L, \Lambda_0)^{\otimes k}} \longrightarrow H(L, \Lambda_0)$$

f_i basis of $H(X, \mathbb{Q})$

e_i " " $H(L, \mathbb{Q})$

$$\langle q^{QP}(f_{i_1} \otimes \cdots \otimes f_{i_\ell}; e_{j_1}, \ldots, e_{j_k}), e_{j_0} \rangle_{PD_L}.$$

$$= \frac{\partial^{k+\ell+1} \Psi}{\partial w_{i_1} \cdots \partial w_{i_\ell} \partial x_{j_1} \cdots \partial x_{j_k} \partial x_{j_0}} (P, Q).$$

symmetrization of $\mathfrak{P}$ of the operator

$$\mathfrak{q} : \underbrace{H(X;\Lambda_0)^{\otimes \ell}}_{\text{symm}} \otimes \underbrace{H(L;\Lambda_0)^{\otimes k}}_{\text{not symm.}} \longrightarrow H(L,\Lambda)$$

$$\mathfrak{q}^{P\omega}(1, e_{i_1} \otimes \cdots \otimes e_{i_k}) = m_k^{P\omega}(e_{i_1}, \ldots, e_{i_k})$$

$A_\infty^{(k,\ell)}$ str. ?

Not well-defined in general.

Case: X toric mfld, L = orbit of T^n action.

$X \circlearrowleft T^n$ $\qquad Q$ T^n-equiv. cycle.

$P = b + x_0[L]^*$, $\quad b \in H^1(L) \otimes \Lambda_0$
$[L]^* \in H^n(L)$.

In this case,
$n(\beta, k, \ell, P, Q) \in \mathbb{Q}$ is well-defined.
(Katz-Liu using S^1-action...)

Suppose Q_1, Q_2 T^n-equiv. cycle,
$Q_1 \sim Q_2 \in H(X)$ but not T^n-equivariantly.

Then, $\Psi(Q_1, P) \neq \Psi(Q_2, P)$. However,

$$P \longmapsto \Psi^{Q_1}(P) = \Psi(Q_1, P).$$
$$P \longmapsto \Psi^{Q_2}(P) = \Psi(Q_2, P).$$

coincide up to change of variables in $H^1(L;\Lambda_0)$

$$\mathfrak{p} = b + x_0 [L]^*.$$

$$\Psi(Q, \mathfrak{p}) = W(Q, b)\, x_0.$$

$$W(Q,b) : H(X; \Lambda_0) \otimes H^1(L; \Lambda_0) \longrightarrow \Lambda_0$$

LG superpotential with bulk. $\quad$ usual Hochschild chain cplx w/ ag.pow.

$$\mathfrak{q}^{\otimes, p} : QH(X;\Lambda)^{\otimes \ell} \longrightarrow CH(H(L;\Lambda_0))$$

$$= \bigoplus_k \mathrm{Hom}(H(L,\Lambda_0)^{\otimes k}, H(L,\Lambda_0))$$

From this, reduce to a map

$$I_{\#}^{\otimes p} : H(X;\Lambda) \longrightarrow HH(HF(L,\Lambda_0))$$

ring hom. $\qquad$ has ring str. & Lie alg. str.

HH $\quad$ Lie + A_∞ alg.

$I_{\#}^{\otimes p}$ respects both, L_∞ homo. $\qquad H(X,\Lambda_0)$ trivial L_∞ alg.

$\cdot \qquad A_\infty$ hom. $\qquad H(X,\Lambda_0)$ $\cup^Q$ + Q-Massey prod.

We consider all L T^n orbits

fibers.

Log fiber(X) : $\quad$ obj is T^n orbits.

Have a map $H(X;\Lambda_0) \longrightarrow \bigoplus_L HH(HF(L))$ ring hom.

(conj: This is isomorphism, under some assumptions (eg CY?))

Now,

$$HH(\mathrm{Lag\,fiber}(X)) \xrightarrow{\text{iso?}} \mathrm{Jac}(\psi^{Q})$$

what is this? Λ_0-module

Def: $\psi^{Q} : H^1(L;\Lambda_0) \to \Lambda_0$

$$\mathrm{Jac}\,\psi^{Q} = \bigoplus_{y \in \mathrm{Crit}\,\psi^{Q}} \frac{Q_y}{\left(\frac{\partial \psi^{Q}}{\partial y_i},\ i=1,\dots,n\right)}$$

change of coords.

Deform q

$$\psi^{Q}(p) = \sum q(Q^{\otimes l}, p^{\otimes k})$$

Thm: (FOOO)

Ring iso : $(H(X,\Lambda_0), \cup^{Q}) \longrightarrow \mathrm{Jac}(\psi^{Q})$.

f_i: basis of $H(X;\mathbb{Q})$.

$$f_i \longmapsto \left[\frac{\partial \psi}{\partial v_i}\right]$$

$$p = \sum x_i e_i$$

$$Q = \sum w_i f_i$$

$$H(X;\Lambda_0) \longmapsto \mathrm{Jac}\,\psi^{Q}$$

$$\psi(Q,P).$$

known: $Q=0$ X is Fano.

PD on $H(X;\Lambda_0)$

Res. on $\mathrm{Jac}(\psi^{Q})$.

Thm2: (OOOF)

$$\langle I_*(v_1), I_*(v_2)\rangle_{\mathrm{Res}} = \langle v_1, v_2\rangle_{PD_M}$$

$v_i \in H(X,\Lambda_0)$

$\psi^{Q} : H^1(L;\Lambda_0) \longrightarrow \Lambda_0$, assume to be <u>Morse</u>

$\mathrm{Jac}\ \psi^{Q} \cong \bigoplus_{y \in \mathrm{Crit}\,\psi^{Q}} \Lambda 1_y$

$\langle 1_y, 1_{y'} \rangle = \begin{cases} 0 & y \neq y' \\ (\det \mathrm{Hess}\, \psi^{Q})^{-1} & y = y'. \end{cases}$

flat structure on universal deformation of ψ^{Q} $\quad \psi^{Q}$

<u>K. Saito</u> $\langle\ \rangle_{\mathrm{Res}}$ $\quad \langle\ ,\ \rangle_{PD_L}$

$I_{\#} : H(X,\Lambda_0) \longrightarrow H(L;\Lambda_0)$.

dualize $I^{\#}$:

$\langle I_{\#}(v), w \rangle_L = \langle v, I^{\#}(w) \rangle_X$.

<u>Thm 3</u>:

$\langle I^{\#}(v_1), I^{\#}(v_2) \rangle_{PD_X}$

$= \sum_{e_I, e_J} g^{IJ} \langle m_2(v_1, e_I), m_2(v_2, e_J) \rangle_{PD_L}$.

e_I is a basis on $H(L)$, $g_{IJ} = \langle e_I, e_J \rangle$.

$m_2 : H(L) \otimes H(L) \longrightarrow H(L)$ (prod in HF?)

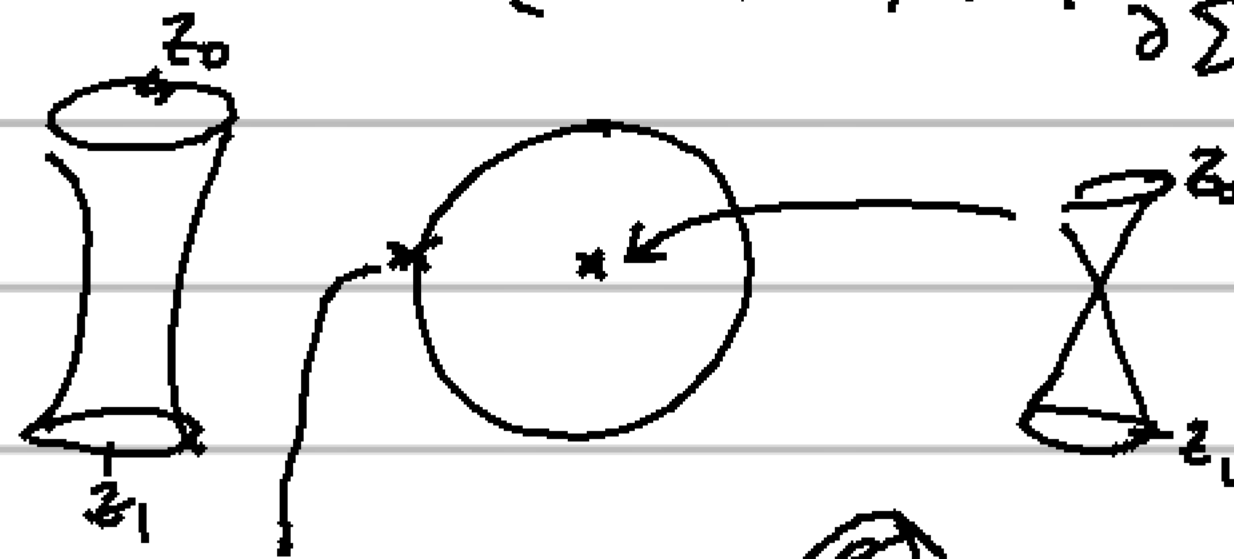
<u>Thm 4</u>: ($\mathrm{Cho} + \varepsilon$)

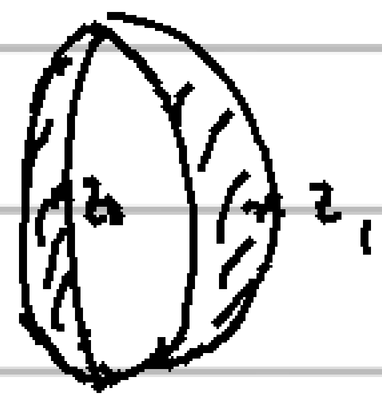
- $(H(L,\Lambda_0), m_2) \cong$ Clifford algebra of $\mathrm{Hess}\, \Psi^{Q}$
- $\cong$ respects PD_L. $\quad$ (Cho: Fano case).

Remaining 8 minutes: Thm 3!

How to do this? mod space of hol. annuli.

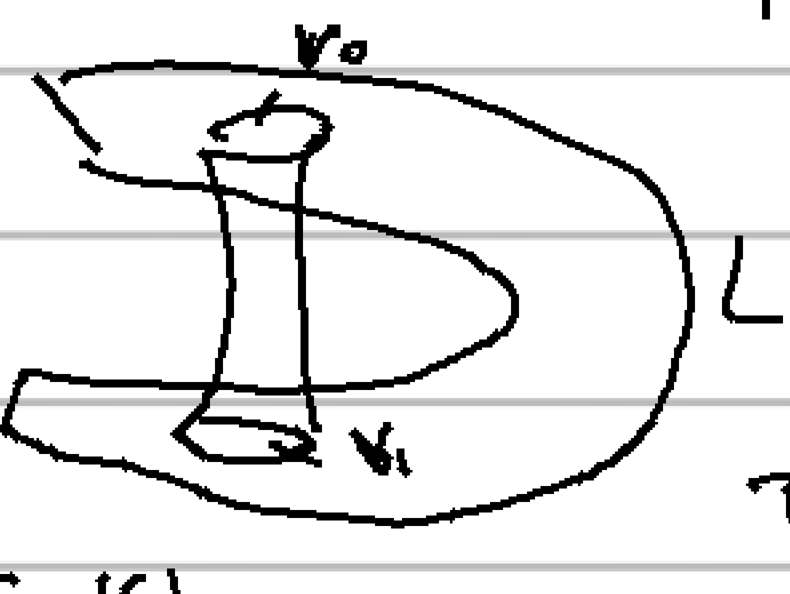
Introduce $\mathcal{J} = \left\{ (\Sigma, z_0, z_1) \;\middle|\; \begin{array}{l} \Sigma \text{ genus } 0, \\ \partial\Sigma = \underset{z_0}{S^1} \amalg \underset{z_1}{S^1} \end{array} \right\}$





Idea: take two singular pts., let them go to each other (like assoc. in quantum cohom.).

$$\mathcal{M} = \left\{ ((\Sigma, z_0, z_1), u) \;\middle|\; \begin{array}{l} (\Sigma, z_0, z_1) \in \mathcal{J} \\ u : (\Sigma, \partial) \longrightarrow (X, L) \\ u(z_0) = v_0 \in L \\ u(z_1) = v_1 \in L \quad \text{fixed} \end{array} \right\}$$



$T_0, T_1 \in \mathcal{J}$

$T_0 =$ 

$T_1 =$ 

Forgetful

$\pi : \mathcal{M} \to \mathcal{J}$

$\# \pi^{-1}(T_0) \rightsquigarrow \sum g^{IJ} \langle m_2(v_0, e_I), m_2(v_1, e_J) \rangle$

And $\# \pi^{-1}(\Gamma_1) = \langle I^\#(v_0), I^\#(v_1)\rangle_X$.

so these two terms, this $\}$, and

$\sum g^{IJ} \langle m_2(v_0, e_I), m_2(v_1, e_J)\rangle$

calculate using Hessian (using toric nature!)