Miami '09 — Kontsevich I

Geom & Alg aspects of WCF

Today: Hyperkähler geometry + other things

lattice $\Gamma \simeq \mathbb{Z}^d$ Γ_u local system

$d = 2s$ $u \in$ parameter space.

For generic u, get a fxn $\Omega : \Gamma_u \longrightarrow \mathbb{Z}$

(doesn't stay the same — there are some walls)

where does this appear?

① Affine structures & asymptotic of SYZ collapse (w/ Yan?)

② Hyperkähler case, integrable systems

③ SUSY string theory, moduli of CY 3-folds (SUSY gauge theory), attractor flows, black holes, —

④ Alg. theory of DT invariants, for 3CY.

Suppose X_ε, family of CY manifolds (any dim.)

J_ε, g_ε CY metric

$\varepsilon \to 0$ maximal degeneration of complex mod!.

$\mathrm{diam}(X_\varepsilon, g_\varepsilon) = 1$.

Th ① $\downarrow$ small tori

B $\dim_{\mathbb{R}} B = n = \dim_{\mathbb{C}} X$

Get a Tropical CY : $B \supset B_{sing}$, codim = 2

$B - B^{sing}$ $\mathbb{Z}$-affine str

$SL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$.

g_B metric on B.

$(g_B)_{ij} = \partial_i \partial_j H$, det = const

(Monge-Ampere?)

Relation to CY metric: semiflat family of non-complete CY.

Canonical torus fiber/$\mathbb{C}$ $\quad (\mathbb{C}^*)^n \quad \dfrac{\log |z_i|}{\log 1/\varepsilon} \quad SL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$

$$\begin{array}{ccc} \downarrow (\varepsilon) & & \downarrow \\ B - B^{sing} & & \mathbb{R}^n \end{array}$$

$\partial \bar\partial \pi^* H$

MS: same B, same g, dual lattice w.r.t. g

$T_{\mathbb{Z}} \subset T_{B - B^{sing}}$

Parameters for $\mathbb{Z}$-aff. structure on $B - B^{sing}$.

$\pi_1(B - B^{sing}) \longrightarrow SL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$

$H^1(B - B^{sing}$, local system on $\mathbb{R}^n)$.

Point of X_ε

- object of $D^b Coh(X_\varepsilon)$
- object of Fuk $(X_\varepsilon^\vee)$ — here can use limiting almost $\mathbb{C}$-str.

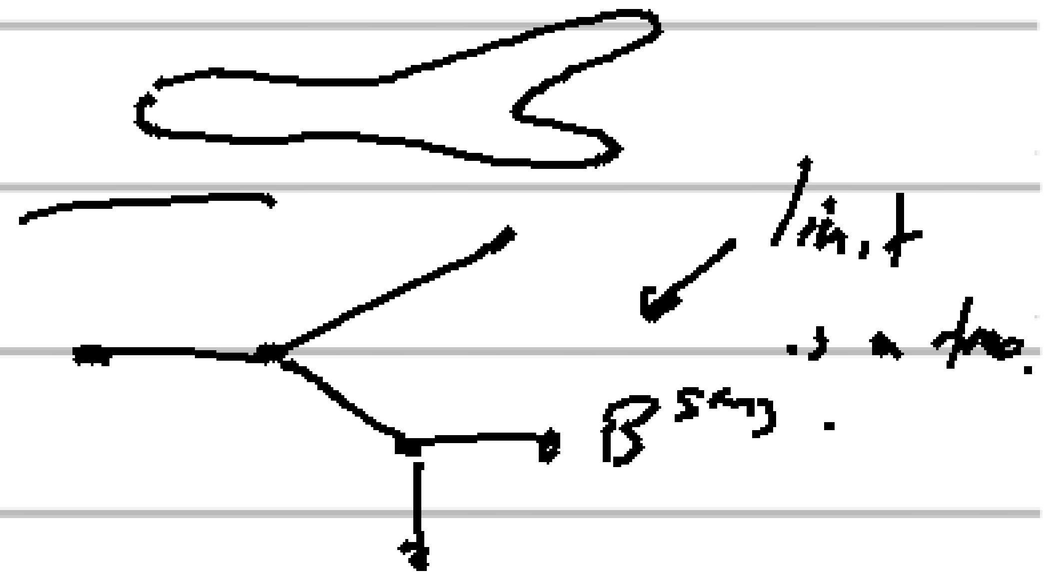
Lag'n mfolds w/ $U(1)$-local systems

toric fibers of $\pi^\vee : X^\vee \to B$

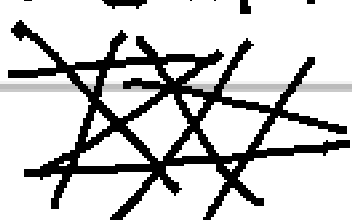
problem - Maslov 0 discs.

T^n

But in limit, these become codim 1 !..



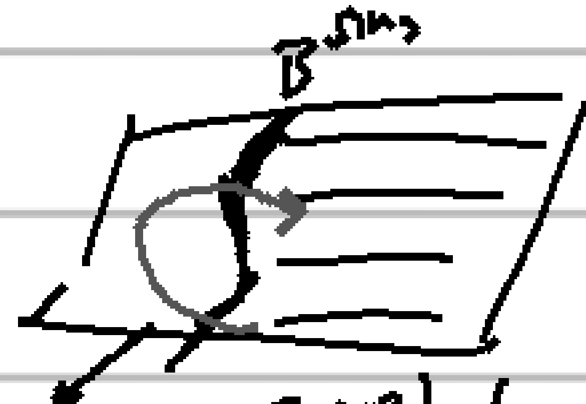
$B - \bigcup$ all trees



coordinate $\circlearrowright (\mathbb{C}^*)^n$ on toric local system.

codim $B^{sing} = 2$

generic orbit B^{sing}



$\mathbb{Z}$ hyperplane foliated by $\mathbb{Z}$ lines,

curved hyperplane in hyperplane

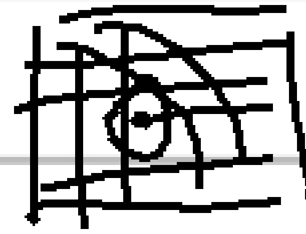
New walls: countably many

2D monodromy:

$x_1 \to x_1(1-x_2)$

$x_i \to x_i$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$



multiple covers glued, so all integer coordinates

$|x_2| >> 1 \quad x \to$

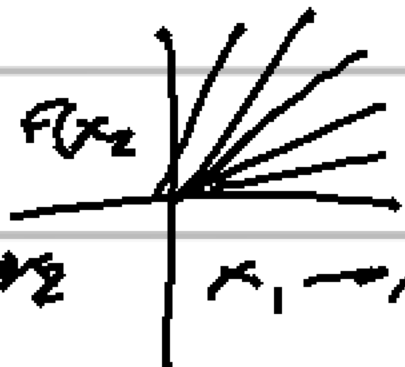
$|x_2| << 1 \quad -x_1 x_2^2 \quad g!$

$\forall$ component of walls:

a hyperplane, foliated by aff. lines.

$x_1 \to x_1 F(x_2)$ $\qquad F(x_2) = 1 + \cdots$

$x_2 \to x_2$ $\qquad$ depends on ε.



$x_1 \to x_1 F(x_2)$

$x_2 \to x_2$

$x_1 \to x_1$

$x_2 \to x_2 F(x_1)$

Group.

$x_1 = x$

$x_2 = y$

$x \to x + \cdots \in \mathbb{C}[[x,y]]$

$y \to y + \cdots \in \mathbb{C}[[x,y]]$

$G = \prod \mathbb{C}\, x^a y^b$, $a+b>0$, $a,b \geq 0$

$= \prod G_\lambda$, $\lambda \in [0, +\infty] \cap \mathbb{Q}P^1$

$\frac{dx \wedge dy}{x \wedge y}$ ⟲ $\mathbb{C}[[x,y]] / \mathbb{C}$.

$\{\,,\}$

$\forall g \in G \;\; \exists!\; g = \prod_{\lambda \uparrow} g_\lambda$ $\qquad g_\lambda \in G_\lambda$

$g_\infty g_0 = g_0 \cdots$

Gross-Siebert generalise to higher dimensions but it's far away from original metric/physics where trees are gradient flows.

we need to solve real Monge-Ampere eq'n.

no explicit formulas in general.

exception:

X_ε - hyperkahler-mfolds.

$\downarrow$

B

integrable systems.

(Y, ω) (complex, symplectic $\omega: \omega^{2,0}$)

$\downarrow$

B

generic fiber Lagrangian abelian variety. (discussion's locus real codim 2)

$\rightsquigarrow \mathbb{Z}$ affine structure.

$\mathrm{Re}\Big(\underbrace{d^{-1}\int_{\gamma_i}\omega^{2,0}}_{\alpha_i}\Big)$ $\mathbb{Z}$-coords. $\dim B = 2n$

$Sp(2n, \mathbb{Z}) \ltimes \mathbb{R}^{2n}$

$\gamma_i \in H_1(\text{fiber}(\mathbb{Z}))$

metric, fibers principal polarized.

$\frac{1}{2\pi\sqrt{-1}}\sum_{i=1}^{n} \alpha_i \wedge \bar{\alpha}_{n+i}$ kahler form on B.

case $Y = K3$

$\downarrow$

$B = \mathbb{CP}^1$.

1-parameter family of affine structures, namely

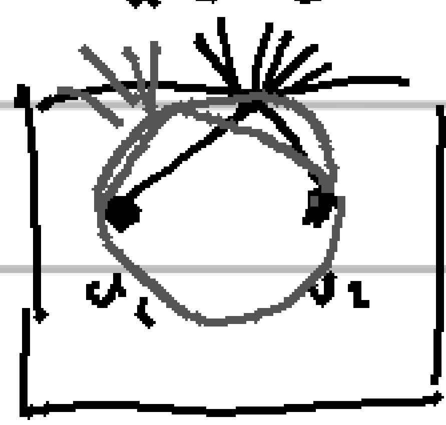
$\omega^{2,0} \longrightarrow e^{i\theta}\omega^{2,0} \rightsquigarrow$ new real hypersurfaces in B.

$\bigcup_{\varphi \in S^1}$ moduli sp. of trees

non-compact example: Seiberg-Witten curve.

$E_u : y + \frac{1}{y} = x^2 - u$ ell. curve. $\omega = \frac{dx\,dy}{y}$

$u \in \mathbb{C}$

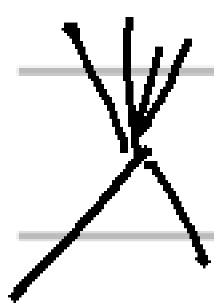


walls for given φ.

now, rotate φ.

Identity in symplect. group

$$T_{a,b} : x \longrightarrow x(1 - x^a y^b)^{-2b}$$
$$y \longrightarrow y(1 - x^a y^b)^{+2a}.$$



$$T_{1,0} \cdot T_{0,1} = T_{0,1} \cdot T_{1,2} \cdot T_{2,3}$$

$$T_{1,1}^{-2} \quad T_{3,2}\, T_{2,1}\, T_{1,0}.$$

$(X\ z)$

$\mathbb{C}_{sym} \quad z \in \mathbb{C}$

$z = 0 \quad \curlyvee$

$(1^{7,14})$

next time:

3 dim CY mfold M

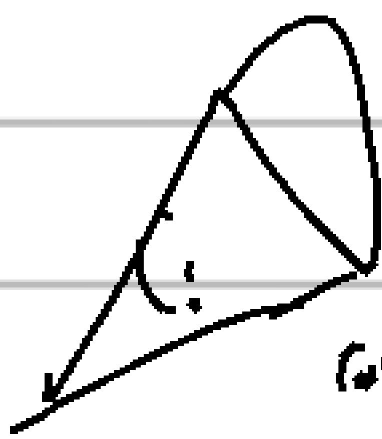
$\mathrm{Mod}_M \leftarrow \qquad \leftarrow$

$\nearrow$ $\mathbb{C}^* \mathrm{bdl.}$ of

mod of $\mathbb{C}$-str.. vol. elts. + bdle of intermediate Jacobians

$H^3(M, \mathbb{C}) / F^2 H^3 + H^3(M, \mathbb{Z})$

pseudo Minkowski $(+ + + -)$ hyperkähler metric



conical bundle, - -