Miami '09 – Kontsevich II

Physics of wall-crossing formulae.

X cpct CY 3-fold

$\mathcal{M}_X$ moduli of $\mathbb{C}$-structures.

$u \in \mathcal{M}_X \rightsquigarrow \Gamma_u = H_3(X_u, \mathbb{Z}) \simeq \mathbb{Z}^{2g} + \text{torsion}.$

$N=2$

$\langle \, , \rangle : \Gamma \otimes \Gamma \to \mathbb{Z}.$

String theory on asymptotically flat space time. at $\infty \sim \mathbb{R}^{3,1} \times \text{CFT}^{N=(2,2)}$ depends on $\mathcal{M}_X \times \mathcal{M}_{X'}$, string coupling constant).

Scattering theory
- particles
- rep. of $\to$ Hilbert space $N=2$ super Poincaré group

$$\mathcal{H} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma$$

BPS inequality

$$\text{Mass}\begin{pmatrix}\text{particle w/} \\ \text{charge } \gamma\end{pmatrix} \geq \left| \int_\gamma \Omega^{3,0}_{X_u} \right|$$

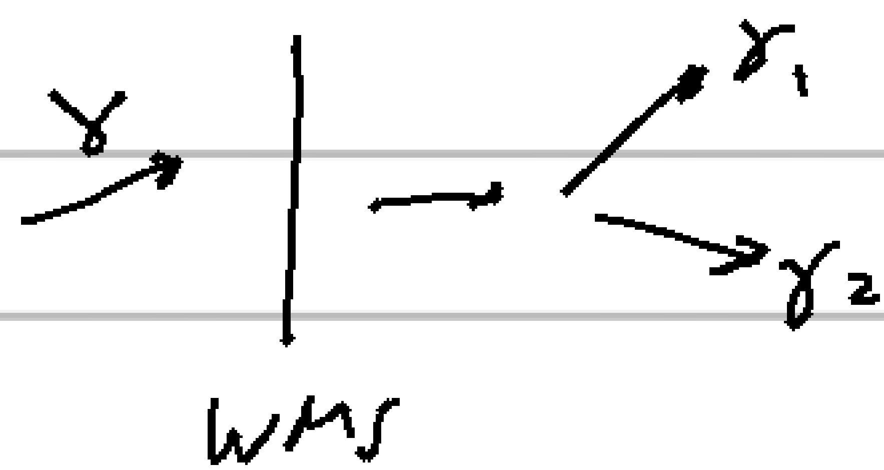
Trace(expression in U(super Poincaré)) = "index" $\Omega_u(\gamma) \in \mathbb{Z}$ should not depend on $\mathcal{M}_X$

$\Omega_u(\gamma)$ change at WMS:

if $\exists\ \gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 + \gamma_2 = \gamma$

s.t. $\int_{\gamma_1} \Omega^{3,0} \parallel \int_{\gamma_2} \Omega^{3,0}$ ∵ γ_1 not parallel to γ_2 but

Denef-Moore formula



γ_1 prim. ch.

$$\Delta \Omega_u(\gamma) = \pm \langle \gamma_1, \gamma_2 \rangle \Omega_u(\gamma_1) \Omega_u(\gamma_2)$$

Choose: $\varepsilon : \Gamma \to \pm 1$ $\quad \dfrac{\varepsilon(\gamma_1 + \gamma_2)}{\varepsilon(\gamma_1)\varepsilon(\gamma_2)} = (-1)^{\langle \gamma_1, \gamma_2 \rangle}$

Torus $\mathrm{Hom}(\Gamma, \mathbb{C}^\times) = (\mathbb{C}^\times)^{2g}$ $\quad \gamma \in \Gamma \rightsquigarrow$ monomial X^γ.

Transformation:

$$T_{\gamma^{-1}} X^{\gamma} \longrightarrow (1 - \varepsilon(\gamma) X^\gamma)^{\langle \gamma, \gamma' \rangle} X^{\gamma'}$$

wall-crossing formula:

choose an angle sector $V \subset \mathbb{C}$ , $< 180^\circ$.

$\rightsquigarrow A_V = \overset{\curvearrowright}{\prod_{\substack{\gamma \in \Gamma \\ \sqrt{\int_\gamma \Omega} \in V}}} T_\gamma^{\Omega_u(\gamma)}$

<u>Axiom</u>: Does not change when we make the point cross ∂V.

$\Omega_u(\gamma) \overset{\text{morally}}{=}$ "counting" of stable objects in $\mathrm{Fuk}(X)$ with stability condition determined by $\mathbb{C}$-structure on X with class

In semiclassical limit → # of special Lag'n mfolds.

Properties of $\Omega_u(\gamma)$:

① wall-crossing formula.

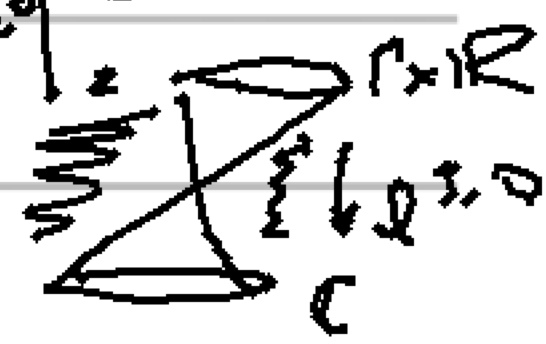
② support property:

for any $u \in \mathcal{M}_X$, $\exists$ norm $\|\ \|_u$ on $\Gamma \otimes \mathbb{R}'$ s.t. if $\Omega_u(\gamma) \neq 0 \longrightarrow \|\gamma\|_u \leq m(\gamma, u)$.

$\gamma, u \in \mathcal{M}_X$

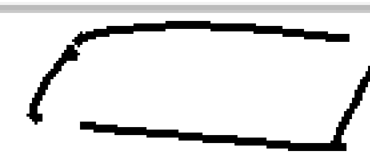
$m(\gamma, u) =$ renormalized mass of particle

$= \dfrac{\left|\int_\gamma \Omega^{3,0}\right|}{\sqrt{\int_X |\Omega^{3,0}|^2}}$.



Suppose we know $\Omega_u(\gamma)\ \forall \gamma$.

WCF $\longrightarrow$ All is determined by values $\Omega_{u_\gamma}(\gamma)$



u_γ attractor point for class γ.

$\exists \lambda \in \mathbb{C}^*$ $\mathrm{Re}[\lambda \Omega^{3,0}_{X_{u_\gamma}}] = \gamma \in H_3(X, \mathbb{R})$

$\tilde{\mathcal{M}}_X \xrightarrow{\text{locally}} H^3(X_u, \mathbb{C}) \xrightarrow{Re} H^3(X, \mathbb{R})$

choose $\longleftrightarrow$ $\Omega^{3,0}$ (alg. coords)

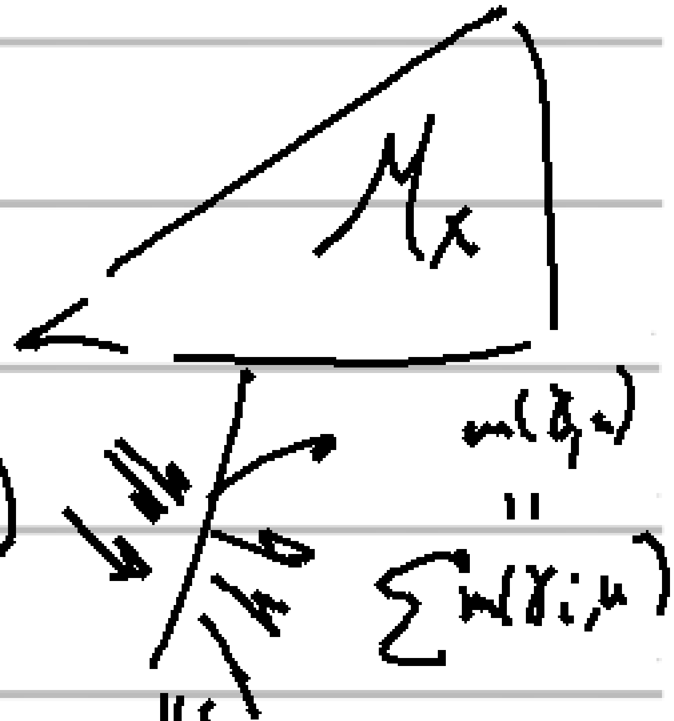
$\mathbb{C}^\times \downarrow$

$\mathcal{M}_X$

$\Omega_u(\gamma) = ?$

$m(\gamma, u)$ function on domain of u, γ, ...

gradient line of $m(\gamma, u)$



Interesting: All critical points are (unclear) ..., of two types: – conifold type $m(\gamma, u) = 0$
– attractor points

Attractor trees — not binary trees in general.

These attractor flows have very simple geometry: rate ...

split attractor flows $\longleftrightarrow$ leaf to $\tilde{\mathcal{M}}_X / \mathbb{R}^+ \to O$

$\boxed{\text{less degenerate picture for disks?}}$

Denef- formula for multi-centered black holes

$\mathbb{R}^3 \times \mathbb{R}^1$

space time

All fibering by w/ base O isotropic symplectic vector space.

asymptotic $\mathbb{R}^3 \setminus (\Sigma)$ black holes $\longrightarrow \mathcal{M}_{X,g}$ $\longleftrightarrow$ harmonic map $\mathbb{R}^3 \setminus \{x_i\} \to \tilde{\mathcal{M}}_X$

$\sum \gamma_i c_i / |x - x_i| + \gamma$ ... ref. pt.

$$\overrightarrow{\prod} \; T_\gamma^{\Omega(\gamma)}$$

domain $Y_\gamma, C^\vee$)

we can glue $\mathbb{C}$ - symp. manifold $\mathbb{C}$ symp. $\downarrow$ standard form.

$$\tilde{\mathcal{M}}_X \underset{u}{\overset{}{\rightleftarrows}} \text{cone in } H^3(X, \mathbb{R}) \supset H^3(X, \mathbb{Z})$$

Wall of second type on $\mathcal{M}_X$. $\exists\, \gamma : \int_\gamma \Omega^{3,0} \in \mathbb{R}_{>0}$

regluing: get a torus bundle, modified at ∞.

upc. $\mathbb{C}$-symp. manifold same thing at ∞

Moduli of vector multiplex

$)($ S^3 $\quad \Omega(\gamma) = 1$.

$\mathcal{M}_\gamma \subset \mathcal{M}_X$

monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ + id.

divisor $\quad \sum x_i^4 = \varepsilon$

OSV conjecture.

u_γ - attractor pt.

$$\Omega_{u_\gamma}(N\gamma) \sim e^{\text{const.}\, N^2 m(u,\gamma)^2}$$

Harvey - Moore. $\quad X^2 \xrightarrow{K3} \mathbb{CP}^1$

(3) New comps of $\mathcal{M}_X$