Miami '09 — Kuznetsov

Fractional CY categories

Let $\mathcal{T}$ be a triangulated cat w/ fin. dim. Hom spaces

Def: A Serre functor to $\mathcal{T}$ is an exact autoequivalence $S_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ s.t. there is a bifunct. iso.

$$\mathrm{Hom}(F, G)^* \xrightarrow{\cong} \mathrm{Hom}(G, S_{\mathcal{T}} F)$$

(Axiomatization of Serre Duality

Ex: $\mathcal{T} = D^b(X)$, X smooth. projective

$\Rightarrow S_X(F) = F \otimes \omega_X [\dim X]$ is a Serre functor.

Def: FCY (Fractional CY) category is a triang cat $\mathcal{T}$ for which $S_{\mathcal{T}}$ exists and $S_{\mathcal{T}}{}^b \cong [a]$ $a, b \in \mathbb{Z}$, $b \neq 0$.

Ex: 1) If X is a CY variety $\Rightarrow$

$$S_X \cong [\dim X]$$

2) If X is an Enriques surface,

$$\omega_X \neq \mathcal{O}_X, \text{ but } \omega_X{}^2 \cong \mathcal{O}_X$$

In $D^b(X)$, $S_X{}^2 = [4]$

3) X is CY, $G : X \circlearrowleft$, action is free, then $D^b(X/G)$.

Def: A <u>semiorthogonal</u> <u>decomposition</u> of a tr. cat. T is a collection $T_1, \dots, T_n$ of str. full subcats s.t.

1) $\mathrm{Hom}(T_i, T_j) = 0$ for $i > j$

2) $\forall F \in T$ $\exists$ chain of maps $F = F_n \to F_{n-1} \to \dots \to F_0 = 0$ s.t. $\mathrm{Cone}(F_i \to F_{i-1}) \in T_i$.

Lemma: If $T = \langle \mathcal{A}, \mathcal{B} \rangle$ is a <u>s.o.d.</u> and T has a Serre functor, then both $\mathcal{A}$ and $\mathcal{B}$ have Serre functors.

Pf: If $F \in T \Longrightarrow$ there is a distinguished Δ

$$\underset{\substack{\uparrow\\ \mathcal{B}}}{F_{\mathcal{B}}} \to F \to \underset{\substack{\uparrow\\ \mathcal{A}}}{F_{\mathcal{A}}} \to F_{\mathcal{B}}[1]$$

$$\left.\begin{aligned} L_{\mathcal{B}}(F) &= F_{\mathcal{A}} \\ R_{\mathcal{A}}(F) &= F_{\mathcal{B}}. \end{aligned}\right\} \text{"mutation" functors}$$

$$\boxed{S_{\mathcal{B}} = R_{\mathcal{A}} \circ S_T, \quad S_{\mathcal{A}}^{-1} = L_{\mathcal{B}} \circ S_T^{-1}}$$

smooth

Theorem: Let $X \subset \mathbb{P}^N$ be a hypersurface of degree $d \leq N$. Then

$$D^b(X) = \langle \mathcal{A}, \mathcal{O}_X, \mathcal{O}_X(1), \dots, \mathcal{O}_X(N-d) \rangle$$

and $\mathcal{A}$ is a FCY of fractional dim. $\dfrac{(N+1)(d-2)}{d}$.

Def: Let Y be a smooth alg. variety, $\mathcal{O}_Y(1)$ be a line bundle.

A Lefschetz decomposition for $D^b(Y)$ is a chain of triang subcats

$$0 \subset \mathcal{A}_{n-1} \subset \mathcal{A}_{n-2} \subset \cdots \subset \mathcal{A}_1 \subset \mathcal{A}_0 \text{ s.t.}$$

$$D^b(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \cdots, \mathcal{A}_{n-1}(n-1) \rangle$$

is a s.o.d.

Ex: 1) $D^b(\mathbb{P}^N) = \langle \mathcal{O}_X, \mathcal{O}_X(1), \cdots, \mathcal{O}_X(N) \rangle$

2) $D^b(\mathbb{P}(w_0, \cdots, w_N)) = \langle \mathcal{O}_X, \mathcal{O}_X(1), \cdots, \mathcal{O}_X(\Sigma w_i - 1) \rangle$

3) $Y = Gr(2, m)$, $m = 2k+1$, $\mathcal{U}$ is the taut. bdle.

$$\mathcal{A}_0 = \cdots = \mathcal{A}_{m-1} = \langle \mathcal{O}, \mathcal{U}^*, \cdots, S^{k-1}\mathcal{U}^* \rangle$$

4) $Y = Gr(2, m)$, $m = 2k$

$$\mathcal{A}_0 = \cdots = \mathcal{A}_{k-1} = \langle \mathcal{O}, \mathcal{U}^*, \cdots, S^{k-1}\mathcal{U}^* \rangle$$

$$\mathcal{A}_k = \cdots = \mathcal{A}_{m-1} = \langle \mathcal{O}, \mathcal{U}^*, \cdots, S^{k-2}\mathcal{U}^* \rangle$$

If $\mathcal{A}_0 = \cdots = \mathcal{A}_{m-1} \Rightarrow$ L.d. is rectangular

Rmk: A L.d. is uniquely determined by $\mathcal{A}_0$.

$$\mathcal{A}_{k+1} = \mathcal{A}_k \cap {}^{\perp}(\mathcal{A}_0(k-1))$$

Theorem: Let Y be smooth & projective, $\dim Y = N$

$\omega_Y = \mathcal{O}_Y(-m)$. $(m > 0)$ and

$$D^b(Y) = \langle \mathcal{A}, \mathcal{A}(1), \cdots, \mathcal{A}(m-1) \rangle$$

(i) Let $X = X_d \xrightarrow{f} Y$ be a smooth divisor in $|dH|$. $d \leq m$.

Then $D^b(X) = \langle \mathcal{B}_d, f^*(\mathcal{A}), \cdots, f^*(\mathcal{A}(m-1-d)) \rangle$.

and $\mathcal{B}_d$ is FCY of fractional dim $\boxed{N+1-2\frac{m}{d}}$.

(ii) Let $X = X_d' \to Y$ be a 2:1 covering, ramified in a smooth divisor in $|2dH|$.

Then $D^b(X) = \langle \mathcal{B}_d', f^*(\mathcal{A}), \ldots, f^*(\mathcal{A}(m-1-d)) \rangle$

and $\mathcal{B}_d'$ is FCY of fr. dim. $\boxed{N+1-\frac{m}{d}}$

$D^b(X) = \langle \mathcal{B}, f^*(\mathcal{A}), \ldots, f^*(\mathcal{A})(m-1-d) \rangle$

$S_{\mathcal{B}}^{-1} = \mathbb{L}_{\langle f^*(\mathcal{A}), \ldots, f^*(\mathcal{A})(m-1-d)\rangle} \circ S_X^{-1}$

$\omega_Y = \mathcal{O}_Y(-m) \Rightarrow \omega_X = \mathcal{O}_X(d-m)$ $\qquad \mathcal{O}_X(m-d)[N-1]$

$= (\mathbb{L}_{f^*(\mathcal{A})} \circ \mathcal{O}_X(1))^{m-d}[N-1]$

$\underbrace{\mathbb{L}_{f^*(\mathcal{A})} \circ \mathcal{O}_X(1)}_{\mathcal{O}}$

$\mathbb{L}_{\mathcal{A}} \circ \phi = \phi \circ \mathbb{L}_{\phi^{-1}(\mathcal{A})}$

$S_{\mathcal{B}}^{-1} \cong \mathcal{O}^{m-d} \circ [N-1]$

$\mathcal{O}^d \cong [2]$

Uses $X \xrightarrow{f} Y \Rightarrow f_*$ is spherical!