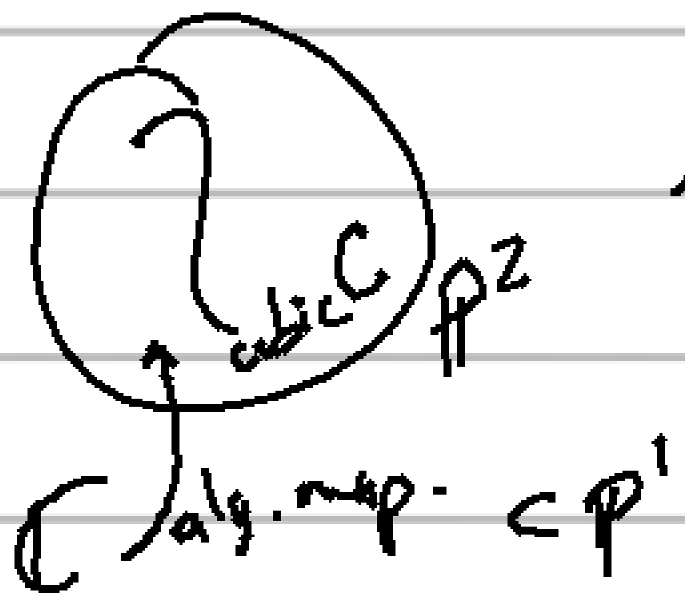
Miami '09: Pandharipande

w/ Gross, Siebert ... /~rahulp/gps.pdf.

Curve counting : $g=0$ in surfaces

K3

or



log CY.

Counting $\mathbb{C} \to \mathbb{P}^2 \setminus C$

$\mathbb{P}^1 \longrightarrow \mathbb{P}^2$

w/ contact only at one pt

count dimensions

$$(1/4) \xrightarrow{d} \mathbb{P}^2$$

$$3d-1 = \dim \overline{M}_0(\mathbb{P}^2, d)$$

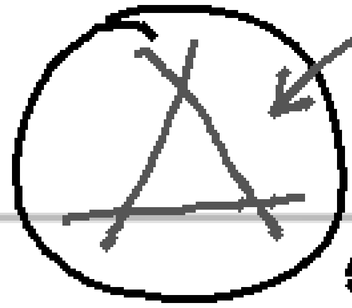
generic deg d. map intersects at $3d$ points, want them all piled up!

conditions are also $3d-1$

$\Rightarrow$ 0 dim'l problem $\longrightarrow$ count #.

N. Takahashi 1999 relates this to $g=0$ invariant of local $\mathbb{CP}^2$.

Ex:



this sort of cubic

$\mathbb{P}^2$

very toric — $\mathbb{P}^2$, divisor both toric.

Can't quite do this (can't have and 3d tangency to a line.)

surface $\to (S, D) \leftarrow$ divisor $\qquad \beta \in H_2(S, \mathbb{Z})$

$\beta \cdot c_1(S) = \beta \cdot D$, so (S, D) is log CY w.r.t. β.

$\Downarrow$ count genus 0 curves w/ full contact at a single pt. of D.

$\mathbb{C}^* \times \mathbb{C}^*$ act basically & group

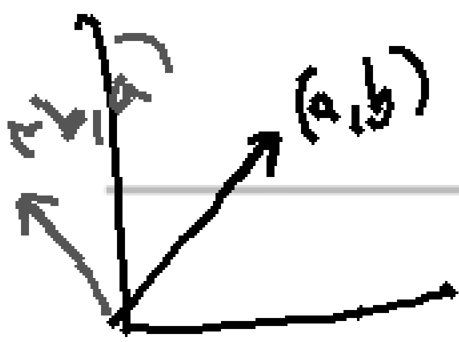
$\mathrm{Aut}^{Gr} = GL(2, \mathbb{Z})$

$\mathrm{Aut}(\mathbb{C}^* \times \mathbb{C}^*)$ (alg. only) $\rightsquigarrow$ not many more (alg. translations)

Formal 1-param families of Aut.

$\mathbb{C}^* \times \mathbb{C}^* = \mathrm{Spec}(\mathbb{C}[x, x^{-1}, y, y^{-1}])$

$A = \mathrm{Aut}_{\mathbb{C}[[t]]}(\mathbb{C}[x, x^{-1}, y, y^{-1}][[t]])$ formal 1-param Auts.



$\mathbb{Z}^2$ $\quad f = 1 + t x^a y^b \cdot g$

where $g \in \mathbb{C}[x^a y^b][[t]]$

Given this, get:

$\theta_{(a,b),f}$: $(x) = x f^{-b}$ $\qquad \theta_{(a,b),f}^{-1} = \theta_{(a,b),f^{-1}}$

$(y) = y f^{a}$

$\in A$.

NB: $(x^a y^b) = x^a y^b$

TVG $\subset$ A (trop. vert. gp.)

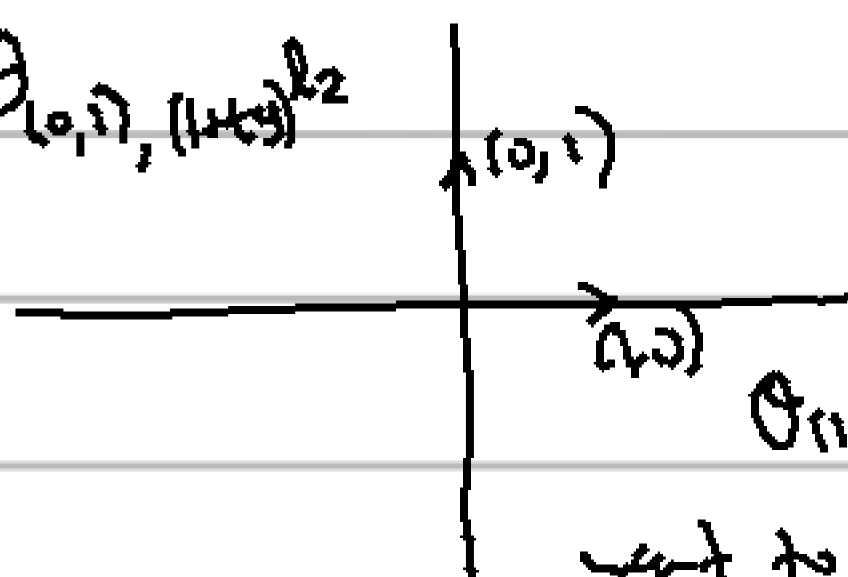
gen. by all such $\theta_{(a,b),f}$

Note that on $\mathbb{C}^* \times \mathbb{C}^*$, w/ $\omega = \frac{dx}{x} \wedge \frac{dy}{y}$, that

$\theta^*_{(a,b),f}(\omega) = \omega$, so

TVG $\subset \mathrm{Aut}^{\mathrm{sympl}} \subset A$.

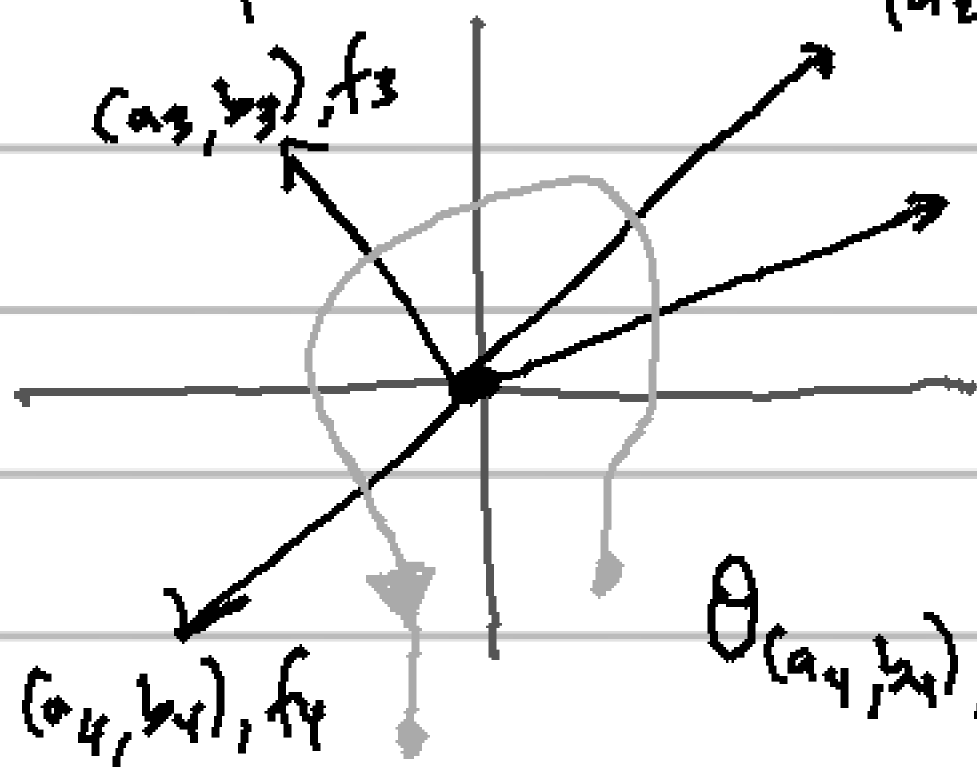
$\theta_{(0,1),(1+ty)^{\ell_2}}$



$\theta_{(1,0),(1+tx)^{\ell_1}}$

want to eventually discuss commutator of these 2 elts.

Path ordered product:

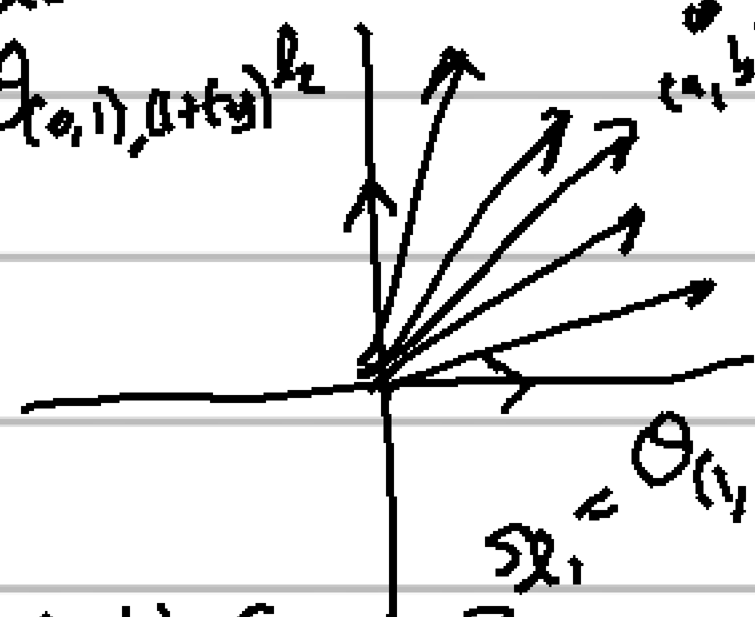


$$\theta_{(a_4,b_4),f_4} \circ \cdots \circ \theta_{(a_1,b_1),f_1}$$

Lemma: (K-S) Given a path ordered product as above, can always add more arrows s.t. resulting path ordered product is the identity.

In our ex:

$T_{\ell_2} = \theta_{(0,1),(1+y)^{\ell_2}}$

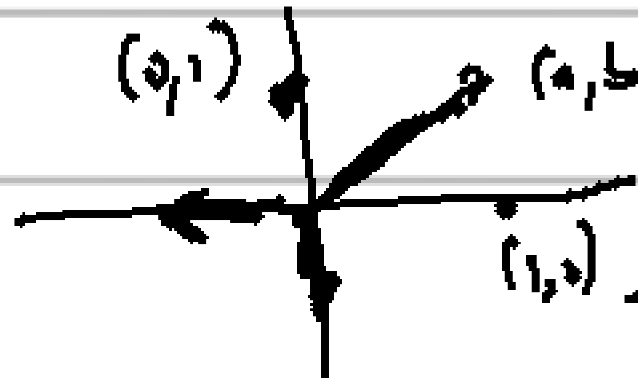


$T_{\ell_2}^{-1} S_{\ell_1}^{-1} T_{\ell_2} S_{\ell_1}$

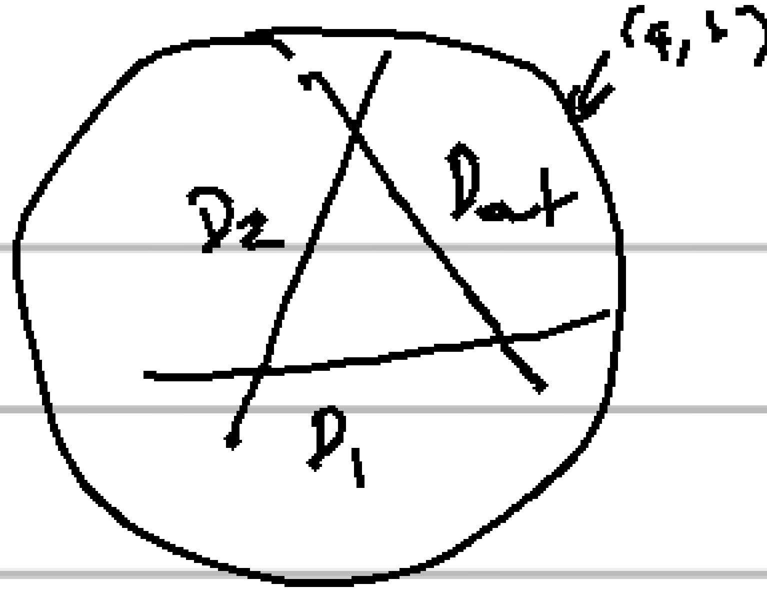
$= \prod$ ← first quadrant

$S_{\ell_1} = \theta_{(1,0),(1+x)^{\ell_1}}$

What is $(a,b)\ f(a,b)$?



3 dark vectors are the fan of a toric surface, weighted $\mathbb{P}^2$!



One knows
$D_1 + D_2 + D_{out} = K_X$
from toric geom.

$f_{a,b}$ is a gen. series for these genus 0 log CY counts associated to exactly this geometry.

Q is ordered partition $Q = (q_1, q_2, \ldots, q_\ell)$

$q_i \geq 0$

$P = (p_1, \ldots, p_{\ell_1})$ $\quad \sum p_i = ak.$

$P' = (p_1', \ldots, p_{\ell_2}')$ $\quad \sum p_i' = bk.$

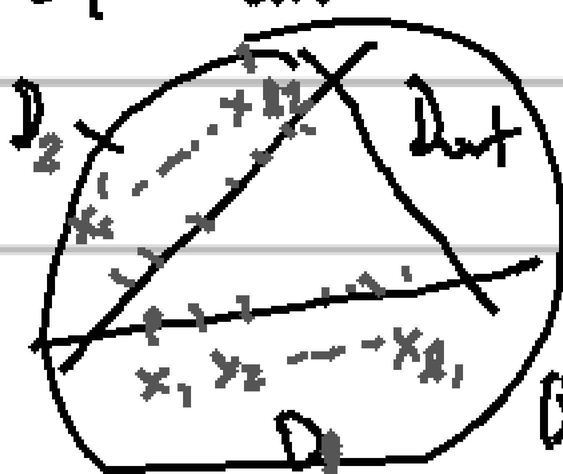
want to define a class $\beta_k \in H_2(\mathbb{P}^2_{(a,b)})$

$\beta_k \cdot D_{out} = k.$ want to only meet D_{out} once, but then might meet others many times.

$\beta_k \cdot D_2 = bk$

$\beta_k \cdot D_1 = ak$

$\tilde{X} \xrightarrow[\pi]{\text{blowup.}}$



$\tilde{\beta}_k = \pi^*\beta_k - \sum p_i E_i - \sum p_i' E_i'$

Now, $(\tilde{X}, \tilde{D}_{out}), \tilde{\beta}_k$ is the usual log CY problem.

So by looking at $(\tilde{X}, \tilde{D}_{\mathrm{wt}})$, $\tilde{\beta}_k$, get

a number $N_{(a,b)}(P_1, P_2)$ ← partitions.

Now, $\log f_{(a,b)} = \sum_{k \geq 1} k\, C^k_{a,b}(l_1, l_2)(t_x)^{ak}(t_y)^{bk}$

$$C^k_{a,b}(l_1, l_2) = \sum_{\substack{\text{Partitions} \\ |P| = ak \\ |P'| = bk}} N_{(a,b)}(P, P') \quad \in \mathbb{Q}.$$

Rmk: before, working over $\mathbb{C}[[t]]$.

For an exact formula, should be working over

$\mathbb{C}[[t_1, \ldots, t_{l_1}, s_1, \ldots, s_{l_2}]]$

then $(1+tx)^{l_1}$ becomes $\prod(1+t_i x)$

$f = \prod(1+t_i x)$

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silence t's now: $f = 1 + a_1 x + a_2 x^2 + \ldots$ $\quad a_i \in \mathbb{Z}$

factor: $f = (1+x)^{a_1}(1+x^2)^{\cdots}(1+x^3) \ldots$

$(1+x^r)$ corresponds to a pt., # r, along curve, gives subscheme of order r, blow up at it, get A_{r-1} singularity, do orbifold GW theory