Miami '09: Pantev

joint w/ Donagi and Simpson

- relationship between non-abel. hodge theory (naht) & geom. langlands correspondence (glc).
- an example. (all / $\mathbb{C}$!)

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1. <u>GLC for GL_n.</u>

everything / $\mathbb{C}$

stacky (D-M stack).

C - smooth, proper (DM) curve

$\mathfrak{Bun}$ = moduli stack of rk. n v.b. on C.

<u>$\mathbf{Bun}$</u> $= \mathfrak{Bun} /\!\!/ \mathbb{C}^\times$ = rigidified moduli stack of bdles (kills generic automorphism).

<u>GLC</u>: Suppose $\mathbb{V} = (V, \nabla)$ algebraic rk. n local system (irreducible), on C.

Then $\exists!$ irreducible D-module $c_{\mathbb{V}}$ on <u>$\mathbf{Bun}$</u>, s.t.

$$H^k(c_{\mathbb{V}}) = c_{\mathbb{V}} \boxtimes \Lambda^k \mathbb{V} \qquad k = 1, \ldots, n$$

(Hecke eigensheaf condition).

H^k - Hecke operators: integral transform.

$$\begin{tikzcd} & \underline{\mathcal{H}ecke}^k \arrow[dl, "p^k"'] \arrow[dr, "q^k"] & \\ \underline{\mathbf{Bun}} & & \underline{\mathbf{Bun}} \times C \end{tikzcd}$$

$\underline{\mathrm{Hecke}}^k = \{(V, V'; \beta)\}$.

V, V' – v. bdles.

$\beta: V \to V'$ – map of loc. free sheaves.

• $\mathrm{supp}(\mathrm{coker}\,\beta) = x \in C$.

length$(\mathrm{coker}\,\beta)$

$H^k = q^k_! p^k_*$

Note: • p^k_x, q^k_x – smooth maps with fibers $Gr(k, n)$.

• One expects that

$$C_V = \mathrm{quant}_{\underline{Bun}}$$

where quant_C = some quantization for coherent sheaves on $T^\vee C$.

$\mathrm{quant}_{\underline{Bun}} = \quad T^\vee \underline{Bun}$.

F-M = Fourier-Mukai on T^*Bun,

2. Non-abelian Hodge Theory.

Non-abelian Hodge thm. for projective w/o ω_C polarization.

Thm (Corlette–Simpson): $(X, \mathcal{O}_X(1))$ – projective

Then, there is an equivalence of $\otimes$-dg cats.

$$\begin{pmatrix}\text{cat. of finite rank} \\ \text{local systems on } X\end{pmatrix} \xrightarrow{\mathrm{nah}_X} \begin{pmatrix}\text{fin. rk } \mathcal{O}_X(1)\text{-semistable} \\ \text{Higgs bdles on } X \text{ w/} \\ ch_1 = ch_2 = 0.\end{pmatrix}$$

Note: ∃ semi-stability condition on one side, but not on other side.
Same as statement that Hodge str. exists b/c of Kahler form, but Hodge decomp. is ind of Kähler class.

Higgs sheaf on $\underline{X}$: $(\mathcal{F}, \varphi)$

$\mathcal{F} \in \mathrm{Coh}(\underline{X})$

$\varphi : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X$

$\varphi \wedge \varphi = 0$

$$\left.\begin{array}{l} \varphi : T_X \otimes \mathcal{F} \to \mathcal{F} \\ S^\bullet T_X \otimes \mathcal{F} \to \mathcal{F} \end{array}\right\} \Leftrightarrow (\mathcal{F}, \varphi) \in \mathrm{Qch}(T^\vee_M).$$

<u>Fact</u>: (Hitchin, Faltings, —) $\mathcal{H}iggs_C \simeq T^\vee \mathrm{Bun}$

stack of all rk. n Higgs bdles.

If $\mathbb{V}$ = rk. n local system

$\rightarrow \mathrm{nah}_C(\mathbb{V})$ = (rank n - Higgs bdle)

$\mathcal{H}iggs_0 = T^\vee \mathrm{Bun}_0$

$\mathrm{quant}_C^{-1}(\mathbb{V}) = \mathcal{O}_{\mathrm{nah}_C(\mathbb{V})}$

$T^\vee \mathrm{Bun}$, $T^\vee \underline{\mathrm{Bun}}$ have natural

cis structure given by the Hitchin map.

$h : T^\vee Bun \longrightarrow B \qquad B = H^0(K_C) \oplus \cdots$

$\underline{h} : T^\vee \underline{Bun} \longrightarrow B \qquad \oplus H^0(K_C)^{\otimes n}.$

$h(E, \theta) = (\mathrm{tr}\,\theta, \ldots, \mathrm{tr}(\theta^n))$

If $\alpha = (\alpha_1, \ldots, \alpha_n) \in B$,

$\overline{C}_\alpha : \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n = 0 \subset T^\vee C.$

Fibers $h^{-1}(\alpha) = \mathcal{P}ic(\overline{C}_\alpha)$. ← Picard stack

$$\underline{h}^{-1}(\alpha) = \underbrace{\mathcal{P}ic}_{\substack{\uparrow \\ \text{rig. defined.}}}(\overline{C}_\alpha) = \underbrace{Pic}_{\substack{\uparrow \\ \text{actual picard variety}}}(\overline{C}_\alpha)$$

If $\overline{C}_\alpha$ is smooth ⟹

we have a poincare sheaf $P \to \mathcal{P}ic(\overline{C}_\alpha) \times \mathcal{P}ic(\overline{C}_\alpha)$.

$$FM : D^b(\mathcal{P}ic) \xrightarrow{FM_P} D^b(\mathcal{P}ic)$$

$$\begin{matrix} \cup & & \cup \\ D^b(\mathcal{P}ic_0) & \xrightarrow[FM]{} & D^b(Pic) \end{matrix}$$

<u>Thm</u>: If $B^{\mathrm{sm}} \subset B$, ⟹ have

$$\mathcal{P} \longrightarrow (T^\vee Bun)^{\mathrm{sm}} \times_{B^{\mathrm{sm}}} (T^\vee Bun)^{\mathrm{sm}}.$$

and $FM = FM_{\mathcal{P}} : D^b_{\mathrm{cmp}}((T^\vee Bun)^{\mathrm{sm}}) \longrightarrow (\qquad)$

$$(T^\vee Bun_0)^{\mathrm{sm}} \overset{FM}{\underset{\sim}{\longrightarrow}} \qquad$$

$\mathrm{FM}\circ\mathrm{quant}_c^{-1}(\mathbb{V}) = \mathrm{FM}(\mathcal{O}_{\mathrm{nah}_c(\mathbb{V})})$.

$$\uparrow$$

$D^b_{\mathrm{comp}}(T^\vee \underline{\mathrm{Bun}})$!

In fact this is a sheaf

$$\mathrm{FM}\circ\mathrm{quant}_c^{-1}(\mathbb{V}) = i_{\alpha *}\mathcal{L}_{\mathbb{V}}$$

$$i_\alpha : \mathrm{Pic}(\overline{C}_\alpha) \hookrightarrow T^\vee \underline{\mathrm{Bun}}$$

$\mathcal{L}_{\mathbb{V}} \in \mathrm{Pic}^0(\mathrm{Pic}(\overline{C}_\alpha))$ corresponding to $\mathrm{nah}(\mathbb{V}) \in \mathrm{Pic}^0(\overline{C}_\alpha)$

Note: $i_{\alpha *}\mathcal{L}_{\mathbb{V}}$ is a Hecke eigensheaf for a classical limit version of the Hecke operators

Note: $i_{\alpha *}\mathcal{L}_{\mathbb{V}}$ is finite over $\underline{\mathrm{Bun}}$

$\Longleftrightarrow$ Higgs sheaf on $\underline{\mathrm{Bun}}$.

Idea: $\mathrm{quant}_{\underline{\mathrm{Bun}}} = \mathrm{nah}_{\underline{\mathrm{Bun}}}$.

Two problems: $\underline{\mathrm{Bun}}$ not a projective variety.

(1) $\underline{\mathrm{Bun}}$ is an Artin stack which is not of finite type.

(2) not proper

Way out: reduce the question to a question about (log) varieties

Thm (Mochizuki):

(X, D) - projective variety + divisor (~~normal crossing~~?)

& X smooth in codim 4

D normal crossings in codim 4.

$\Rightarrow$ $\begin{pmatrix}\text{tame parabolic}\\ \text{local systems on}\\ (X,D)\end{pmatrix} \xrightarrow{\mathrm{Sch}_{X,D}} \begin{pmatrix}\text{locally abelian}\\ \text{tame, semistable}\\ \text{parabolic Higgs}\\ \text{bdles on } (X,D)\end{pmatrix}$

w/ par $ch_1 = 0$, par $ch_2 = 0$

locally abelian — means near every pt. of divisor where normal crossing, parab Higgs bdle $\cong \oplus$ parabolic line bdles

Thm: (Mochizuki): If V - quasi-projective.

$$Y \xhookleftarrow{f} V \xhookrightarrow{\varphi} X$$ - compactifications

and $D = X \setminus V$ (X, D) - as before

Then there is an equivalence

$\begin{pmatrix}\text{semisimple parabolic tame}\\ \text{local sys on } (X,D)\end{pmatrix} \longrightarrow \begin{pmatrix}\text{semisimple } \mathcal{D}\text{-modules on } Y\\ \text{which are smooth on } V\end{pmatrix}$

want to apply this to $\mathcal{B}_{\tilde{u}}^{ss}$

$\swarrow$ resolve

$\underline{\mathcal{B}_u}$ $\mathcal{B}_u^{ss}$

$\underline{\mathcal{B}_u}^{vs}$ $\nwarrow$ moduli stacks of very stable bdles.