Miami '09 — Thomas I:

Curve Counting and Derived Categories

Stable Pairs

1. GW, MNOP, virtual cycles, stable pairs
2. wall crossing
3. ~~GV, BPS~~

X smooth proj. CY 3-fold, i.e. $K_X = \mathcal{O}_X$
(can extend to all 3-folds. . .)

All "things" live in families of virtual dim 0, critical pts. of a function.

SLags, vbles, curves, sheaves, surfaces

Hope to define invariants by "counting" them.

- compactness of moduli space (stops pts. $\to \infty$) — finite invts — deformation invariance
- transversality / virtual cycles

These lectures – count hol. curves in X

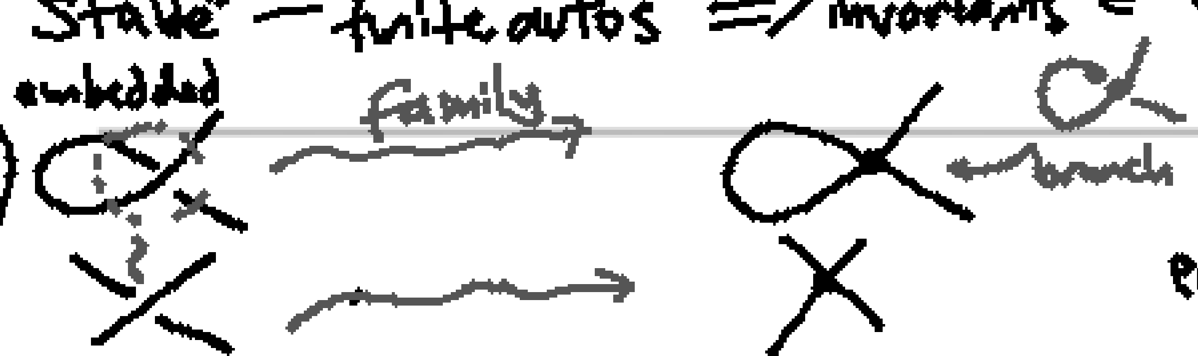
GW theory counts stable maps:

|| keep curve nice, map bad

nodal curve $\longrightarrow X$

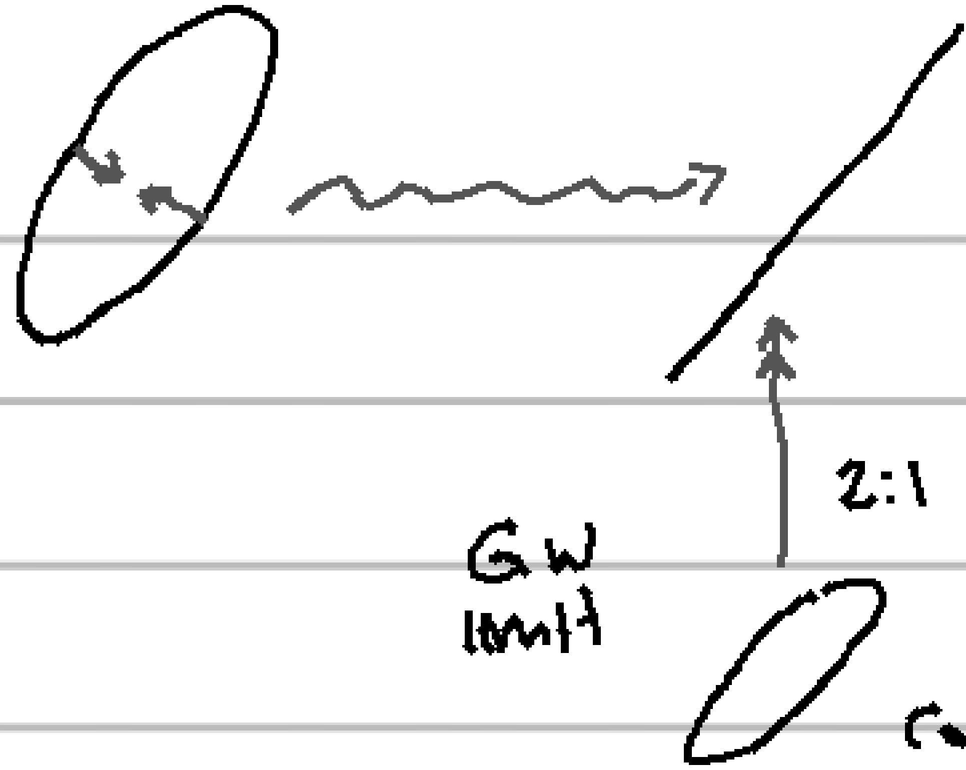
"Stable" — finite autos $\Rightarrow$ invariants $\in \mathbb{Q}$.

A)



In GW theory limit is just normalization map — not embedded at crossing pt.

(B)



conic ⇝ line

GW limit

2:1 double cover

$\mathbb{Z}/2$ - autom.

counts as $1/2$ to GW-invt.

(Mohammed: how does this count as $1/2$ if before as 1? Look at families — marked pts.)

(2) Subschemes

MNOP theory

Hilbert scheme(X)

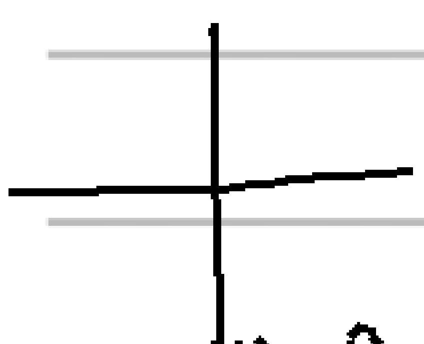
cpct.

Map embedding curve bad.

(A)

embedded pt.

Local model.



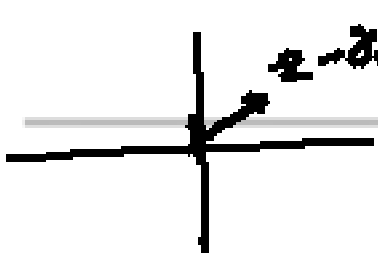
$x=0=z$

$y=0,\ z=t \to 0.$

$$(x, z)\cdot(y, z-t)$$

$$=(xy, yz, x(z-t), z(z-t))$$

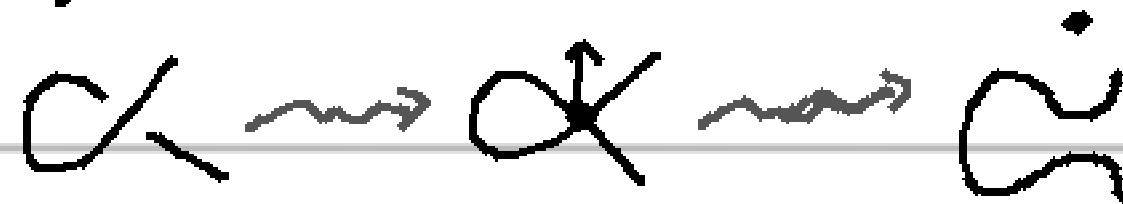
$t \to 0$



$$(xy, z) \supset (xy, yz, xz, z^2)$$

$z \notin$ ideal, but $z\cdot$ any elt. of max ideal is $\in$ ideal.

Genus change,
free pts.



$g \mathrel{+}= 1$

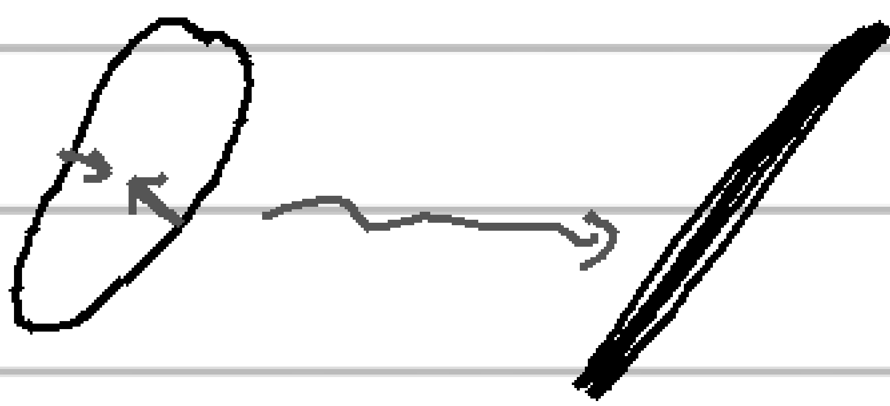
$\#$free pts $\mathrel{+}= 1$

Subschemes contain zero-dim. subscheme.

No autos! $\Rightarrow$ Invts. $\in \mathbb{Z}$

(bound $\#$free pts. by chern class)

(B)



thickened line

$x^2 = 0 = z$

MNOP Conjecture: $Z_{GW}(u) = \sum_g N^\bullet_{g,\beta} u^{2g-2}$

$\beta \in H_2(X)$

disconnected GW theory

other side:

$n = \chi(\mathcal{O}_Z) = 1 - g\,(C) + \#\,(\text{free pts.})$

subscheme $Z = C \cup$ free pts.

$Z_{MNOP,\beta}(t) = \sum_n I_{n,\beta}\, t^n$ $\quad \#^{vir}(\mathrm{Hilb}^n X)$

$Z_{MNOP,0}(t) = \sum_n I_{n,0}\, t^n$ deals w/ free pts

$$\boxed{\frac{Z_{MNOP,\beta}(t)}{Z_{MNOP,0}(t)} = Z_{GW,\beta}(u), \quad -e^{-iu} = t}$$

rat'l fxn, invt. under $t \to 1/t$.

Some sort of analy. contin. going on.

e.g.

$$t(1+t)^{-2} = t - 2t^2 + 3t^3 - 4t^4 + \cdots$$

inv't. under $t \leftrightarrow 1/t$

Laurent expansion, NOT inv't.

Stable pairs

$(\mathcal{F}, s)$ ← $H^0(\mathcal{F})$

sheaf, supp. in codim 2 = dim 1.

$[\mathcal{F}] = \beta$.

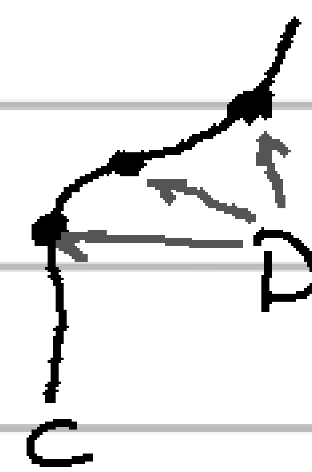
satisfying: $\begin{cases} \bullet\ \mathcal{F} \text{ is pure} \\ \bullet\ s \text{ has 0-dim'l coker.} \end{cases}$

no 0-dim'l subscheme

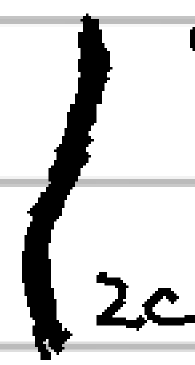
Examples:

- $(\mathcal{O}_C, 1)$
- $(\mathcal{O}_C(D), s_D)$

  curve + free pts. on C.



- $(\mathcal{O}_{2C}, 1)$ limit of picture (B)



$\mathcal{O}_{C_1 \cup C_2} \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}$

(A)

stable pair limit

$(\mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}, (1,1))$

coker $\mathcal{O}_p$

$(\mathcal{O}_{C'}(p), s_p)$

pure $\Rightarrow$ C is Cohen-Macaulay, i.e.:

- no embedded pts.
- no free pts.

(when C is Gorenstein, stable pair $\Longleftrightarrow$ (C, D) $D \subset C$ is a pt. of $\mathrm{Hilb}\, C$)

why Gorenstein? need ω_C:

Aside: $\mathrm{Ext}^1_C(Q, \mathcal{O}_C)$ $H^0(Q \otimes \omega_C)^*$

coker(pair)

$$n = \chi(F) = 1 - g(C) + \#(\text{free pts.})$$

$$Z_{P,\beta} = \sum_n P_{n,\beta}\, t^n$$

Conjecture: $$Z_{P,\beta}(t) = \frac{Z_{MNOP,\beta}(t)}{Z_{MNOP,0}(t)} = Z_{GW,\beta}(u)$$

and again, this is the Laurent expansion of a rat'l function that's invariant under $t \longleftrightarrow 1/t$.

Note: Over a good component,

$$(\mathcal{O}_C, 1) \xrightarrow{\text{equiv. data}} \mathcal{I}_C \qquad (\mathcal{I}_C \to \mathcal{O}_X \to F \to Q)$$

for good C ideal sheaf