Miami '09: Thomas II

<u>Self-dual obstruction theory.</u> (Behrend)

<u>Model</u>: $f: \mathcal{A} \xrightarrow{\text{smooth}} \mathbb{C}$

$\mathcal{M} = (df)^{-1}(0)$ $\quad\mathcal{M}$ has vdim 0!

Suppose $\dim \mathcal{M} > 0$, $\mathcal{M}$ smooth.

$$0 \longrightarrow T\mathcal{M} \longrightarrow T\mathcal{A}|_{\mathcal{M}} \xrightarrow{D(\partial f)} T^*\mathcal{A}|_{\mathcal{M}} \longrightarrow \text{coker} \longrightarrow 0$$

Hessian $(\partial^2_{ij} f)$, symmetric

Dual map $T^*\mathcal{A}|_{\mathcal{M}} \longleftarrow T\mathcal{A}|_{\mathcal{M}}$ is also $D(\partial f)$

coker $= T^*\mathcal{M}$.

Deformations/obstructions are dual (def'n of self-dual obs. theory).

Perturb f by a fn. g on $\mathcal{M}$. $d(f+\varepsilon g)^{-1}(0)$

$=$ zeros of dg on $\mathcal{M}$.

Virtual cycle should be $c_{\dim \mathcal{M}}(T^*\mathcal{M}) = (-1)^{\dim \mathcal{M}} e(\mathcal{M})$

In general ($\mathcal{M}$ self-dual obs. theory, not necessarily smooth), Behrend shows that $[\mathcal{M}]^{\text{vir}} \in H_0(\mathcal{M})$ is

$$e(\mathcal{M}, \chi^B) = \sum_{n \in \mathbb{Z}} n\, e((\chi^B)^{-1}(n))$$ for some constructible fn.: $\mathcal{M} \xrightarrow{\chi^B} \mathbb{Z}$.

$\chi^B \equiv (-1)^{\dim M}$ where M is smooth

Depends only on the local analytic str. of M!

For simplicity, ignore χ^B, signs, just work w/ e. [very powerful, can't compute vir. fund. class this way]

E.g. MNOP — do def. theory not of subscheme $Z \subset X$, not of ideal sheaf $\mathcal{I}_Z$, but of sheaf $\mathcal{I}_Z$ triv. determinant.

$$T_Z I_n(X,\beta) = \mathrm{Ext}(\mathcal{I}_Z, \mathcal{I}_Z)_0$$
$$\mathrm{obs}_Z = \mathrm{Ext}^2(\mathcal{I}_Z, \mathcal{I}_Z)_0$$

Serre dual (CY^3)

Stable pairs: def. theory not as a pair, but of the trivial determinant 2-term $I^\bullet \in D^b(X)$.

$$I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\} \in D^b(X)$$

$$T_{I^\bullet} P_n(X,\beta) = \mathrm{Ext}^1(I^\bullet, I^\bullet)_0$$
$$\mathrm{obs}_{I^\bullet} = \mathrm{Ext}^2(I^\bullet, I^\bullet)_0$$

dual

Work with a fixed CM curve $C \subset X$. $\chi(\mathcal{O}_C) = 1-g$

$$I_n(X,C) \subset I_{1-g+n}(X,\beta)$$

locus of subschemes whose largest CM subscheme is C.

$Z = C \cup$ embedded pts. / free pts.

Similarly,
$P_n(X, C) \subset P_{n+1-g}(X, \beta)$
pairs supp. on C

$$\frac{Z_{MNOP,\beta}}{Z_{MNOP,0}} = Z_{P,\beta}$$

follows from (restricting to curve)

$I_{n,C} = P_{n,C} + P_{n-1,C}\, e(X) + P_{n-2,C} \cdot e(\mathrm{Hilb}^2 X) + \cdots + e(\mathrm{Hilb}^n X)$

curve in pts.

n pts. in C

$n-1$ pts in C, 1 free pt on X.

$e(I_n(X,C))$ etc.

<u>Wall crossing</u> : $Z = C \cup \{P_i\}$

$$\mathcal{I}_Z \to \mathcal{I}_C \to \mathcal{O}_{P_i}$$

$$\mathcal{O}_{P_i}[-1] \to \mathcal{I}_Z \to \mathcal{I}_C. \quad (*)$$

Move across a wall in space of stab conditions, so that "slope" "phase" of $\mathcal{O}_P[-1]$ crosses that of $\mathcal{I}_C$.

When bigger, $(*)$ destabilizes $\mathcal{I}_Z \notin \{\text{stable objects}\}$

However, extensions $\mathcal{I}_C \to ?? \to \mathcal{O}_{P_i}[-1]$ are now stable.

(Can work out w/ LES that $\mathrm{Ext}^1(\mathcal{O}_{P_i}[-1], \mathcal{I}_C) = \mathrm{Ext}^1(\mathcal{O}_{P_i}, \mathcal{O}_C)$

Exts ?? are of the form.

$I^\bullet = \{\mathcal{O}_X \to F\}$

$\mathcal{I}_C \xrightarrow{\ker}$, $\xrightarrow{\mathrm{coker}} \mathcal{O}_{P_i}[-1]$

$\updownarrow$

$\mathcal{O}_C \to F \to \mathcal{O}_{P_i}$.

{stable objects} changes as we cross wall
from $I_n(X,\beta)$ to $P_n(X,\beta)$.

Invariants change?

<u>Model</u>: Sheaves E which can become unstable as cross a wall in Kähler cone

$$0 \to A \to E \to B \to 0$$

slope A crosses slope B.

but A, B do not decompose, remain stable on both sides of wall.

$$\Rightarrow \mathrm{Hom}^{\leq 0}(A,B) = 0 = \mathrm{Hom}^{\leq 0}(B,A).$$

So by Serre Duality, $\mathrm{Ext}^{\geq 3}(A,B) = 0$ etc.

So only have:

$$\mathrm{Ext}^1(A,B) : 0 \to B \to F \to A \to 0 \quad ②$$

$$\mathrm{Ext}^1(B,A) = \mathrm{Ext}^2(A,B) : 0 \to A \to E \to B \to 0 \quad ①$$

When we cross a wall, lose all extensions ①, $\mathbb{P}(\mathrm{Ext}^1(B,A))$

gain all extensions ② $\mathbb{P}(\mathrm{Ext}^1(A,B))$

$e = \mathrm{ext}^1(B,A)$ $\qquad e = \mathrm{ext}^2(B,A)$

$$\text{Difference} \overset{\text{R.R.}}{=} \chi(B,A)$$
$$= -\chi(A,B)$$

So Difference in invariants as cross wall is $\chi(B,A)\, e(M_A) e(M_B)$.

Our case: $A = \mathcal{O}_{p_i}[-1] \leftarrow$ split up, semistable autos.

$B = \mathcal{I}_C \leftarrow$ stable

Our case: bunch of strings.

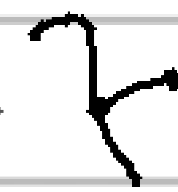
1 free pt.

$$I_{1,c} = P_{1,c} + e(X)$$

Let's see this using geometry in case C smooth.

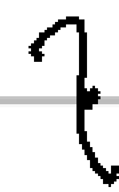
$p \in X \setminus C \rightsquigarrow$ $+1$ to $I_{1,c}$

$\qquad$ 0 to $P_{1,c}$



$p \in C \rightsquigarrow$ $2 = e(\mathbb{P}^1)$ to $I_{1,c}$

$\qquad$ 1 to $P_{1,c}$



See that $\Rightarrow I_{1,c} = e(X \setminus C) + \underbrace{2}_{e(\mathbb{P}^1)} e(C)$

$$P_{1,c} = e(C)$$

$$I_{1,c} = e(X) + P_1(c)$$

$$I_{1,c} - P_{1,c} = e(X)$$

General C, $p \in X$

contributes $e(\mathbb{P}(\underbrace{\mathrm{Hom}(I_C, \mathcal{O}_p)}_{\hom(I_C, \mathcal{O}_p)}))$ to $I_{c,1}$

$e(\mathbb{P}(\underbrace{\mathrm{Ext}^1(\mathcal{O}_p, \mathcal{O}_C)}_{\mathrm{ext}^1(I_C, \mathcal{O}_p)}))$ to $P_{c,1}$

By Riem. Roch, $\hom(I_C, \mathcal{O}_p) - \mathrm{ext}^1(I_C, \mathcal{O}_p) = 1$.

~~Diffs~~ in inv'ts is $+1$ $\forall p \in X$.

$\int_X \Rightarrow$ Diff. in inv'ts is $e(X)$.