Miami '09 : Thomas III

C CM curve $\subset X^3$ CY

T 0-dim sheaf

$e\left(\mathrm{surj}(I_C, T) / \mathrm{Aut}\, T \right) t^{[T]} \rightsquigarrow$ contribs of δ_C^{1} to

length(T) (T stack of 0-dim sheaves)

$e(\mathbb{P}\mathrm{ure}\,\mathrm{Ext}^1(T, \mathcal{O}_C)/\mathrm{Aut}\, T)\, f^{(T)}$ L mnop (

$\rightsquigarrow Z_P^C(f)$

Exts don't factor through quotient of T.

$\underline{[T] = 2}$

$$\frac{\mathcal{O}_p \oplus \mathcal{O}_q}{} \quad \mathrm{Hom}(I_C, \mathcal{O}_p \oplus \mathcal{O}_q) - \mathrm{Hom}(I_C, \mathcal{O}_p) - \mathrm{Hom}(I_C, \mathcal{O}_q) + \{0\}$$

$\mathrm{Aut}\ \mathcal{O}_p \oplus \mathcal{O}_q$

$$\frac{\mathrm{Hom}^*(I_C, \mathcal{O}_p)}{\mathbb{C}^*} \times \frac{\mathrm{Hom}^*(I_C, \mathcal{O}_q)}{\mathbb{C}^*} = \mathbb{P}(\mathrm{Hom}_p) \times \mathbb{P}(\mathrm{Hom}_q)$$

Similarly, pairs $\rightsquigarrow e_p e_q$

Difference $\underline{e_p + e_q + 1}$

$$\mathcal{O}_p \oplus \mathcal{O}_p \quad \frac{\mathrm{Hom}(I_C, \mathcal{O}_p)^{\oplus 2} - \text{rk } 1 \text{ Homs}}{\mathrm{Aut}(\mathcal{O}_p \oplus \mathcal{O}_p) = GL_2(\mathbb{C})}$$

$\mathrm{Hom}(\mathbb{C}^2, \mathrm{Hom}(I_C, \mathcal{O}_p))$

$\mathrm{Gr}(2, \mathrm{Hom}(I_C, \mathcal{O}_p))$

$e = \binom{h_p}{2}$

Pairs $e = \binom{e_p}{2}$ Difference e_p $\mathcal{O}_x \oplus \mathcal{O}_p$

$\int_{X \times X^{[p]}} = P_{1,c}\, e(X) = P =$

$\mathcal{O}_{2p}$ $\quad \underline{\mathrm{Hom}(I_C, \mathcal{O}_{2p})} - \underline{\mathrm{Hom}(I_C, m_p)}$ $\quad \mathcal{O}_p \oplus \mathcal{O}_q$ $\quad e_p + e_q = 1$

Aut

$e = h_{2p} - h_{m_p}$ $\quad$ Pairs $e_{2p} - e_p$

difference I.

$\int_{J^2}$ ← integrate over all J of len 2.

So $I_{2,C} - P_{2,C} = P_{1,C}\, e(X) + e(\mathrm{Hilb}^2 X)$ $\quad e(\mathrm{Hilb}^2 X)$

$e(X/G) \neq e(X)/e(G)$, no good if $e(G) = 0$

Use virtual Poincaré polynomials

$$\mathbb{P}^n = \frac{\mathbb{C}^{n+1} - \{0\}}{\mathbb{C}^*} = \frac{q^{n+1}-1}{q-1} = q^n + q^{n-1} + \cdots + 1 \xrightarrow{q \to 1} n+1.$$

$P_q(\mathbb{C}) = q$

$\lim_{q \to 1} P_q = e.$

Maxim: call them Serre polynomials, not Poincaré

Exercise: $P_q(Gr(k,n)) \xrightarrow[q\to 1]{} \binom{n}{k}$

$$P_q(GL(n,\mathbb{C})) \underset{q\sim 1}{\sim} n!\,(q-1)^n$$

(the max. [illegible] of tori

Joyce "virtual indecomposability"

$$T \in \mathcal{J} \quad Surj(I_c, T) = \underbrace{(Hom(I_c,T)}_{q^{n_T}} - \bigcup_{T_1\subset T} Hom(I_c,T_1) + \bigcup_{T_2\subset T_1\subset T} Hom(I_c, T_2)$$

Divide by Aut T_1, take P_q.

$$PureExt(T, \mathcal{O}_c) = Ext^1 \longrightarrow \bigcup_{T_1\subset T} Ext^1(\mathcal{O}_c, T/T_1)$$

Reader sum

$$\bigcup_T Hom(I_c,T)\, t^{[T]} \Big/ Aut\, T - \bigcup_{T_1,T_2\in T} Hom(I_c,T_2)\, Ext(T_2,T_1)\, t^{[T_1+T_2]} + \cdots$$

$Aut(T_1) + Aut(T_2)$

Aut of $T_1 \to T \to T_2$ —— $(1 + Hom(T_2,T_1))$

Good b/c get $q^{\mathrm{ext}'(T_2,T_1)-\hom(T_2,T_1)}$

Can use RR, SD; same as $q^{\mathrm{ext}'(T_1,T_2)-\hom(T_1,T_2)}$

<u>Hall algebra</u> Ringel, Joyce, Toen, KS, Bridgeland, Toda, Nagao...

$$1_{\mathcal{J}} * 1_{\mathcal{J}} = \{\text{Extending filtrations } T_1 \to T \to T_2\}$$

$$\downarrow \mathcal{J} \ (T)$$

$$\mathcal{J}$$

Extend to any

$$A \to \mathcal{J} \leftarrow B \qquad A \times B$$

e.g. $\hom(I_c, T)$ over $T \in \mathcal{J}$ $\to$ $q^{\hom(I_c,T)}$.

$$\begin{array}{ccc} A \times B & \to & 1_{\mathcal{J}} * 1_{\mathcal{J}} \to \mathcal{J} \\ \downarrow & & \downarrow (T_1, T_2) \\ A \times B & \to & \mathcal{J} \times \mathcal{J} \end{array}$$

$T_0 \in \mathcal{J}$ can form $1_{T_0} \to \mathcal{J}$

$$\mathrm{Surj}(I_c, \cdot) * 1_J \longrightarrow J$$
$$\| $$
$$\mathrm{Hom}(I_c, \cdot) \qquad \phi : I_c \longrightarrow J$$

is surjection $I_c \to (\mathrm{im}\ \phi)$
followed by extension
$\mathrm{ind} \subseteq J \to$ on ϕ

$$1_J = 1_0 + 1_{J^*}$$

0-sheaf

non-zero sheaves

(duality for $*$)

$$1_J^{-1} = 1_0 - 1_{J^*} + 1_{J^*} * 1_{J^*} - 1_{J^*}^{*3} + \cdots$$