Miami '09: Zharkov

Geometry of numbers & tropical curves

$V \cong \mathbb{R}^g$

$Q : \Gamma_2^* \xrightarrow{\sim} \Gamma_1$ (polarization)

$\Gamma_1, \Gamma_2 \subset V$
$\cong \; \cong$
$\mathbb{Z}^g$

view Q as bilinear form on V^*

symmetric & > 0

$X = V/\Gamma$, tropical ppAV.

$$\Theta(x) = \max_{\lambda \in \Gamma_2^*} \{\lambda \cdot x - \tfrac{1}{2}\lambda\, Q(\lambda)\}$$

$$V \longrightarrow \mathbb{R}$$

(Γ_2 is like integral lattice, gives integral structure
Γ_1 period lattice, almost arbitrary, needs to be compat. w/ Γ_2 in some sense.)

PL convex fcn.

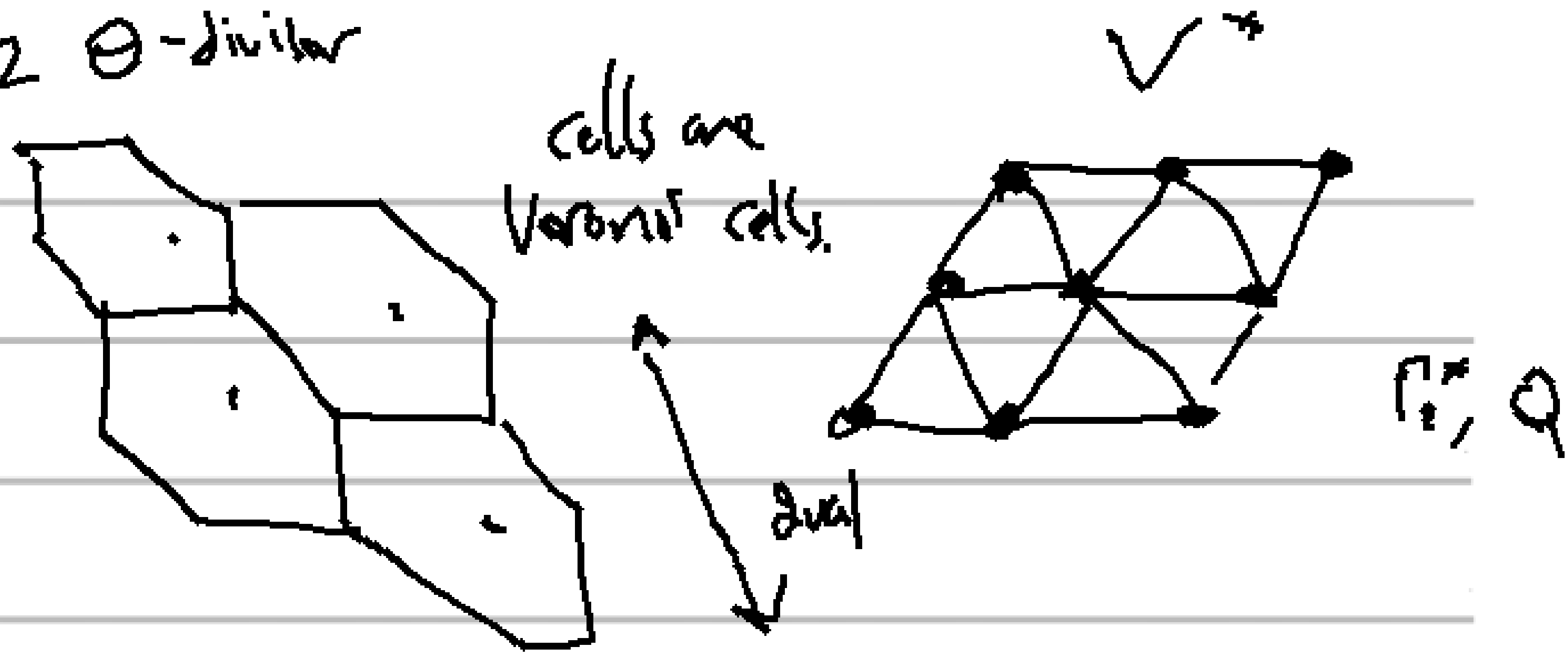
Θ = corner locus of $\Theta(x)$

Θ-divisor.

Ex. of what this looks like in dim. 2

$$\Theta(x+\gamma) = \Theta(x) + \underbrace{Q^{-1}(\gamma)\cdot x + \tfrac{1}{2} Q(\gamma)}\;\; \gamma$$

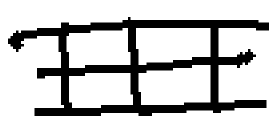
$g=2$ Θ-divisor



Delaunay decomposition.

A dicing is a Delaunay given by families of hyperplanes through lattice points of $\Gamma_{\mathbb{Z}}^*$

(in dim $g=2,3$, Every Delaunay decomp. is a dicing)

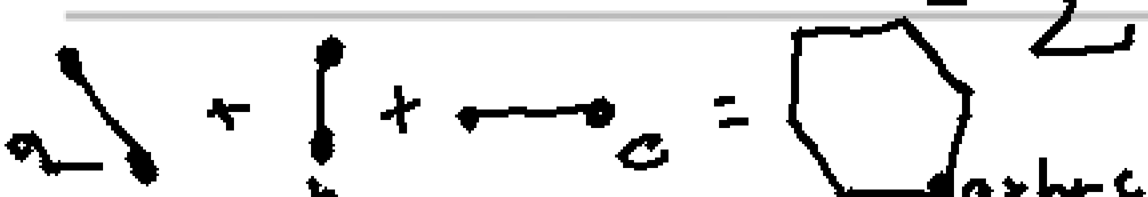


Maximal $\Longleftrightarrow$ no more possible hyperplanes

$l_1, \dots, l_r \in T_{\mathbb{Z}}$ primitive normal vectors, TFAE:

- $\{l_1, \dots, l_r\}$ totally unimodular. $r \geq g$
- $Q = \sum \alpha_i (l_i)^2, \ \alpha_i > 0$
- Voronoi cell = Zonotope (Minkowski sum of intervals)

$$= \sum [\alpha_i l_i]$$



$g > 5$ no classification of diagrams (?).

Example $D^1(g)$ ~ decomposition of  into

$g!$ simplices

$$0 \le x_{\sigma(1)} \le x_{\sigma(2)} \le \dots x_{\sigma(g)} \le 1$$

Delaunay

$l_1, \dots, l_g$, $l_{ij} = l_i - l_j$, $i < j$ $\quad \sigma \in S_g$

$\frac{g(g+1)}{2}$ zone vectors.

zonotope = g-permutahedron.

C ~ metric graph (no 1-valent vertices)

Ex: 

$g = b_1(C)$

E = {edges w/ fixed orientation}.

1-forms

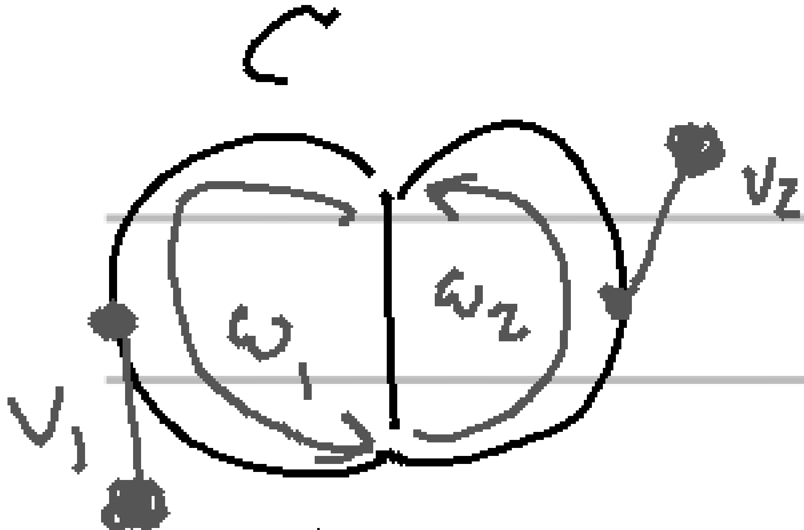
" Γ_2^* = lattice of $\mathbb{Z}$-circuits (Kirchhoff ?)

$\Gamma_2 = \{v_e\}$

v_e = evaluation of circuits at edge e.

Γ_1 = lattice of $\mathbb{Z}$-cycles

$\delta_2(\omega_1) = -b$ $\quad Q_2 : \Gamma_2^* \xrightarrow{\sim} \Gamma_1$.

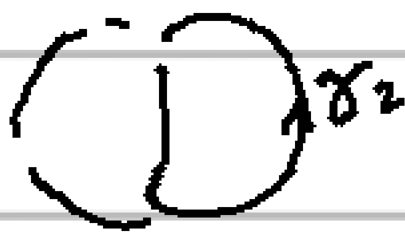


$\Gamma_2^* = \{w_1, w_2\}$

$\Gamma_2 = \{v_1, v_2\}$

$\Gamma_1 = \{\gamma_1, \gamma_2\}$

$$\Gamma_1 = \left\{ \begin{pmatrix} a+b \\ -b \end{pmatrix} \quad \begin{pmatrix} -b \\ b+c \end{pmatrix} \right\}.$$



$$Q = \sum_{e \in E} (\text{length } e)\, v_e^2$$

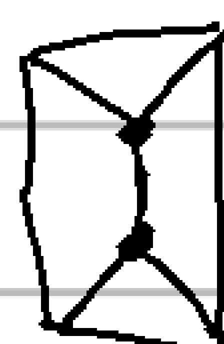
v_e – zone vectors.

Schottky –

$g = 4$

$3g-3$ zone vectors

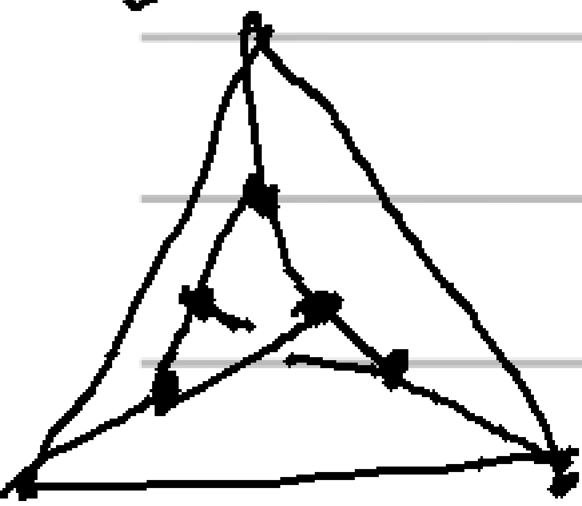
$\frac{g(g+1)}{2}$ dim M_{PPAV}



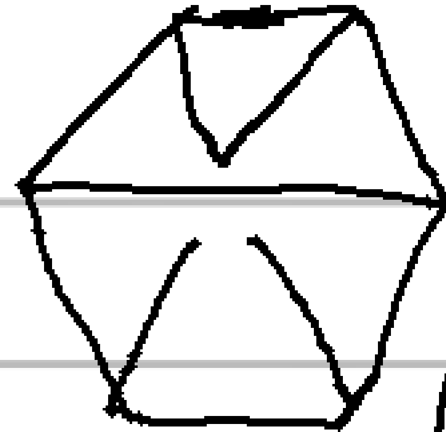
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \end{pmatrix} \subset D^1(4)$$

missing $\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$

$g=5$:



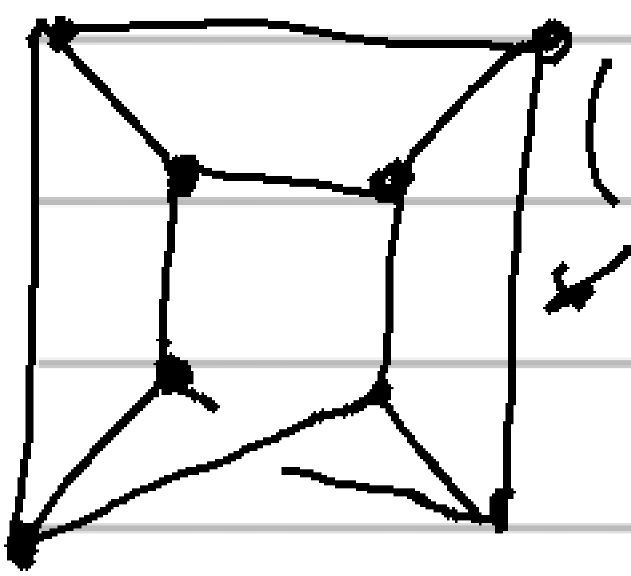
$$\begin{pmatrix} 1&0&0&0&1&0&1 \\ 1&1&0&0&0&0&1 \\ 0&1&1&0&0&1&1 \\ 0&0&1&1&1&1&1 \\ 0&0&0&1&1&1&1 \end{pmatrix} \qquad (1) \mapsto \begin{pmatrix} 1\\1\\0\\0\\1 \end{pmatrix}$$



$$\begin{pmatrix} 1&1&0&0&1 \\ 0&0&1&1&1 \\ 1&0&1&0&1 \\ 0&1&0&1&1 \end{pmatrix}$$

maximal!

If not the last, not maximal!



(eh.)

Another tiling, not from a graph!

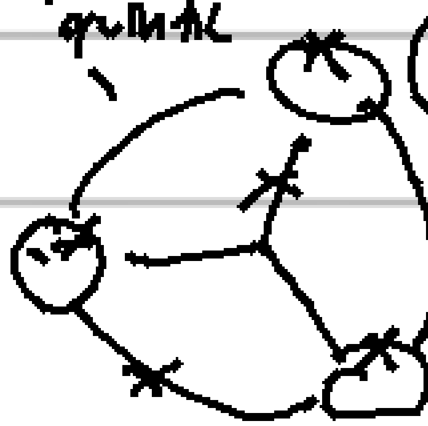
$$\begin{pmatrix} 1&&&&&1&0&1&0&1 \\ &1&&&&0&1&0&1&1 \\ &&1&&&1&1&0&0&1 \\ &&&1&&0&0&1&1&1 \\ &&&&1&1&1&1&1&1 \end{pmatrix}$$

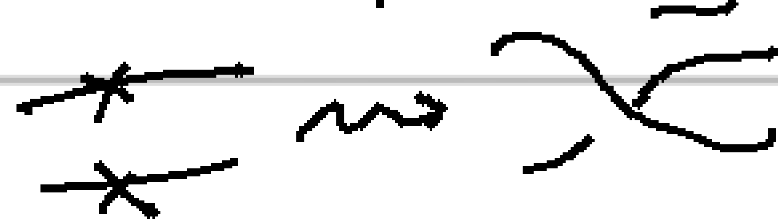
Does not come from graph (trivalent graph has 12 vertices?)

Hope:

2 copies

plane quintic





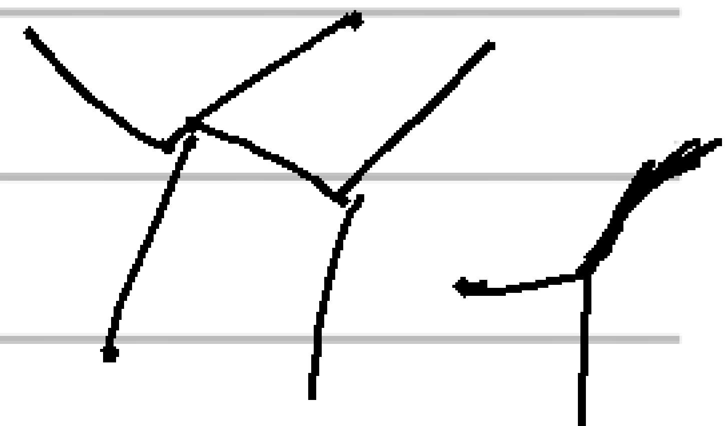
Get curve of $g = 11$, with τ involution.

$\mathrm{Prym}(\tilde{C}, \tau) = (V^{*-})^* / \Gamma_1^-$

dim 5 — circuits on graph which are anti-invariant w.r.t. involution.

On X

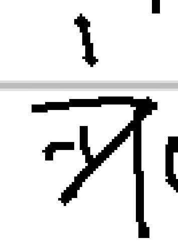
A_k = tropical k-cycles



$\sum w_i p_i$

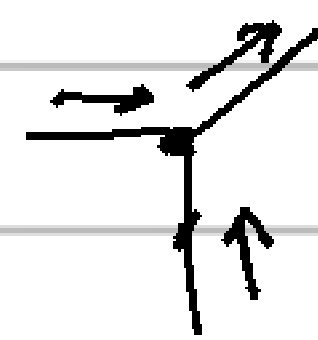
$\|$

$Z \in A_k$

or 

$\downarrow$

$[Z] \in \Lambda^k \Gamma_2 \otimes \Lambda^k \Gamma_1$



$k=1$

$\sum w_i \det\langle p_i \rangle \otimes p_i$

$H_k(Y, \Lambda^k \Gamma_2)$ — data this sum is a cycle. — cw cell — $w \in \Lambda \Gamma_2^*$

$\mathrm{Hdg}_k = \Lambda^k \Gamma_2 \otimes \Lambda^k \Gamma_1 \cap \ker \varphi$ — space of Hodge classes

$\varphi : \Lambda^k V \otimes \Lambda^k V \longrightarrow \Lambda^{k-1} V \otimes \Lambda^{k+1} V$

$u_1 \wedge \cdots \wedge u_k \otimes v_1 - v_k \longrightarrow \sum (-1)^i u_1 \cdots \hat{u}_i \cdots \otimes v_1 - v_k.$

$\text{Hodge}: A_k \longrightarrow (*)g_k.$

\$1,000,000

Hodge is not surjective for $D^3(5)$

Given this tropical version, construct complex family to show actual hodge fails

$$X_C = (\mathbb{C}^*)^g / e^{(?, \frac{1}{2}}.$$