

Toen - Saturated dg-Categories, I

1/18/2010

* Present some "recent" results on saturated dg-algebras.

Origin of saturated dg-algebras:

* X smooth proper scheme/ k

$\mathcal{D}_{\text{perf}}(X)$. \exists a generator $E \in \mathcal{D}_{\text{perf}}(X)$ (known res. H).

$\mathcal{B} := \mathcal{R}\underline{\text{End}}(E)$ dg-algebra/ k .

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$\underline{\text{End}}^*(E')$, E' an injective replacement of E .

Fact: • $\mathcal{D}_{\text{perf}}(X) \simeq \mathcal{D}_{\text{perf}}(\mathcal{B}\text{-dgmod})$

• \mathcal{B} is saturated

** X a CW complex, ΩX (topological group).

$\pi_0(X) = *$.

$C_*(\Omega X, k) =: \mathcal{B}$ dg algebra/ k .

multiplication in \mathcal{B} is induced by $\Omega X * \Omega X \rightarrow \Omega X$.

- When X is a finite CW,

\mathcal{B} is smooth ($1/2$ saturated).

- $\mathcal{D}(\mathcal{B}\text{-dgmod}) \simeq \mathcal{D}_{\text{loc}}(X, k) \subseteq \mathcal{D}(X, k)$

"
 $\{E \mid H^i(E) \text{ is locally constant } \forall i\}$

$\mathcal{D}(X) \simeq \mathcal{D}(\mathcal{B})$. Many interesting invariants of X can be recovered from \mathcal{B} .

ex. - X sm. proper scheme/ k , $\text{char } k = 0$

$HH_*(\mathcal{B}/k) \simeq \bigoplus H^*(X, \Omega_{X/k}^*)$ (Hodge cohomology)

- X finite CW cplx,

$HH_*(\mathcal{B}/k) \simeq H_*(\mathcal{L}X)$, $\mathcal{L}X = \text{Map}(S^1, X)$.

Part I: Finiteness results about dg algebras.

k a comm. ring (not necessarily char 0), everything over k .

Def: B is a dg alg. / k . (associative with unit).

$D(B)$ = derived cat. of B -dg modules.

* B is proper if B is a perfect complex of k -modules.

(i.e. $B \xrightarrow[\text{q. iso.}]{} \text{bounded complex of proj. } k\text{-mod of finite type}$)

$\Leftrightarrow B$ is compact in $D(k)$.

Remark: k field, B proper $\Leftrightarrow H^*(B)$ is finite dimensional
 \swarrow (need to derive when k is not a field)

* B is smooth if B is compact in $D(B \overset{(L)}{\otimes}_k B^{op})$.

(i.e. $[B, \oplus \cdot] \simeq \oplus [B, -]$).

("B. is of fin. homol. dim").

* B is saturated if it is smooth + proper.

* B is of finite type \forall filtered system of dg-algebras B_α , we have:

$$[B, \underset{\alpha}{\text{colim}} B_\alpha] \simeq \underset{\alpha}{\text{colim}} [B, B_\alpha]$$

where $[B, c] = \text{set of morphisms in } H_0(\text{dg-alg}/k)$

(Remark: system is filtered, so

homolim = colim.
actually this statement is w/ homolim).

$$H_0(\text{dg-alg.}/k) := (\text{quasi-isom})^{-1}(\text{dg-alg}/k).$$

(happy analogue of being finitely presented as an algebra).

Prop: - finite type \Rightarrow smooth

- saturated \Rightarrow finite type.

(examples of dgas that are smooth but not finite type exist! e.g. A^∞ -infinite pts.?)

Examples: (*) X scheme, flat of fin. type $/k$. Then: ^{(\Rightarrow separated),}

$$D_{\text{qcoh}}(X) \simeq D(B)$$

- B proper $\Leftrightarrow X$ is proper.
- B smooth $/k \Leftrightarrow X$ smooth $/k$.
- B finite type $\Leftrightarrow X$ smooth $/k$.

(**) ~~finite CW~~ X finite CW, $D_{\text{ac}}(X, k) \simeq D(B)$

$$B = ({}_*(S^2 X)).$$

B is of finite type (thus smooth). Almost never proper.

(***) Q a finite quiver.

$B = A(Q, k)$ (path algebra) of finite type $/k$.

Remark: We will see that these notions (sm, proper, f. type) all have a "nice" categorical interpretation in terms of duality in a certain \otimes -2 cat of dg-cats. (saturated = most dualizable, the others just being $1/2$ dualizable).

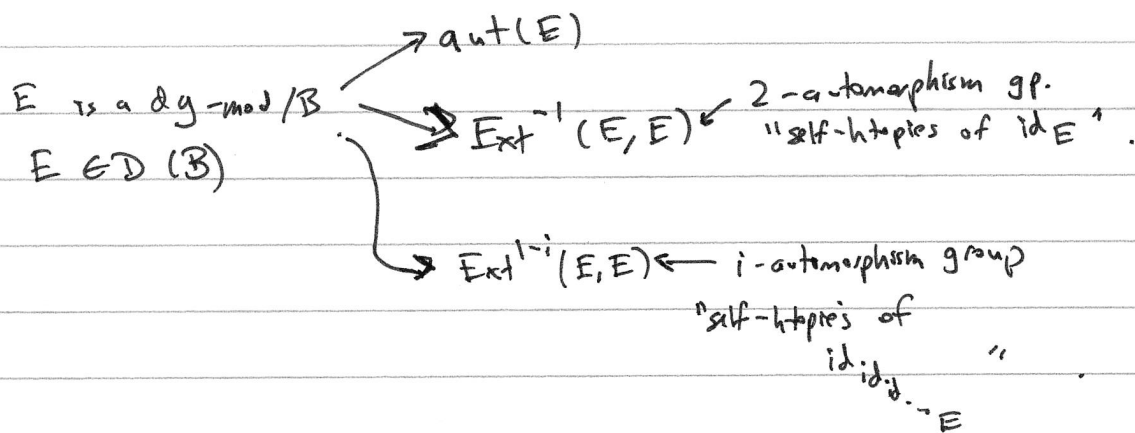
Main theorems:

- (1) If B is a finite type dg-algebra $/k$, then there exists an algebraic moduli space of finite-dimensional B -dgmodules
- (2) There is a moduli space for saturated dg-algebras (up to quasi-isom.) (not up to Morita equiv, which would be the right notion)
(false up to Morita equiv. \Rightarrow only formal stack for Morita, can't be alg.)

Remark: These moduli spaces are ?

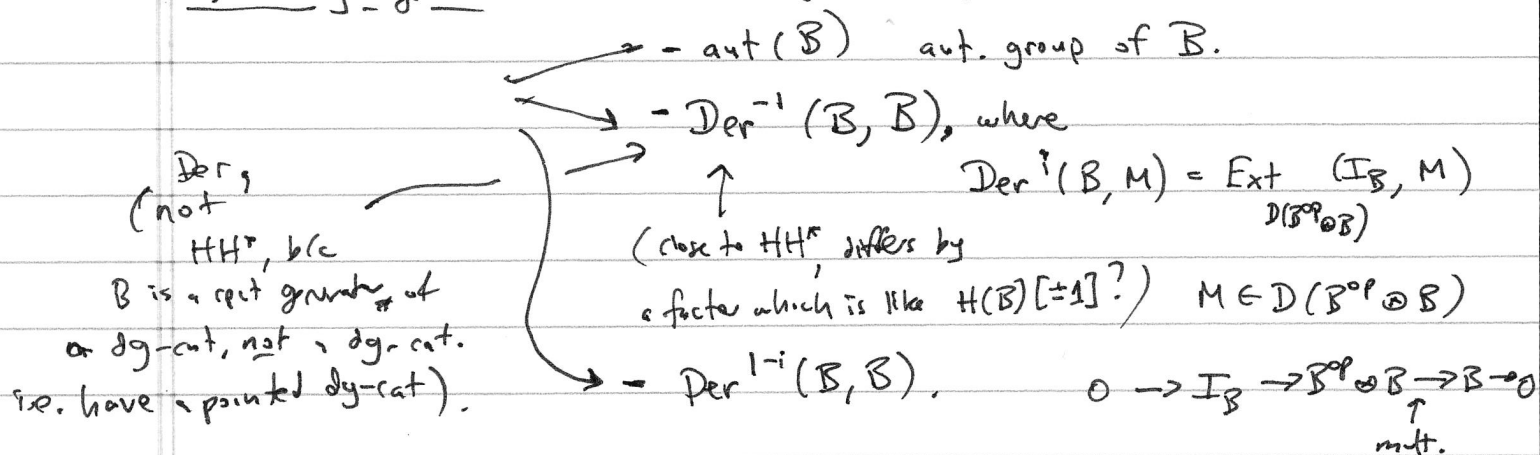
- They should be, at least, algebraic stacks, because objects have automorphisms.
- In both cases, there are "higher automorphism groups":

\mathbb{E}



Conclusion: Moduli spaces in (1) of them must be algebraic ∞ -stacks.
(In the same way that alg 1-stacks have underlying schemes up to a stratification, these ∞ -stacks do too).

Case of dg algebras : $B \in H_0(\text{dg-alg}/k)$.



Consequences:

(*) Up to stratification, we get schemes of finite type / k classifying saturated dg-algebras and fin.-dim'l dg-modules over a fin.-type dg-alg, (+ $\text{aut}(E), \text{Ext}^{1-i}(E, E)$, -- are flat group schemes locally defined on those schemes).

(**) B fin.-type, then \exists an alg. space, fin.-type / k M_B , classifying simple objects, i.e. $M_B \longleftrightarrow \left\{ E \in D(B) \mid \begin{array}{l} \text{fin.-dim'l} \\ \text{Ext}^{-i}(E, E) = 0 \\ \text{Ext}^0(E, E) \simeq k \end{array} \right\}$

M_B comes equipped w/ a natural class $\alpha \in H_{\text{et}}^2(M_B, G_m)$, obstruction for existence of a universal simple dg-module on M_B .

$[\text{torsion} \xleftrightarrow{\alpha} \exists \text{ an "Azumaya alg." somewhere}]$.

Another
Application:

X is a compact complex manifold.

If $D_{\text{coh}}^b(X) \simeq D_{\text{def}}(B)$, B saturated, then X is algebraic.

Rmk on how to do this: write X as sub-plx. manifold of said moduli space associated to B , do some more work.