Miami Mirror Symmetry Conference

Toen – Saturated dg-Categories, I 1/18/2010

* Present some "recent" results on saturated dg-algebras.

Origin of saturated dg algebras:

* X smooth proper scheme/k

$D_{perf}(X)$. $\exists$ a generator $E \in D_{perf}(X)$ (known result).

$B := R\underline{End}(E)$ dg-algebra/k.

$\quad\quad \shortparallel$

$\underline{End}^*(E')$, E' an injective replacement of E.

Fact: • $D_{perf}(X) \simeq D_{perf}(B\text{-dgmod})$

• B is saturated

** X a CW complex, ΩX (topological group).

$\pi_0(X) = *$. $C_*(\Omega X, k) =: B$ dg algebra/k.

multiplication in B is induced by $\Omega X \times \Omega X \to \Omega X$.

- When X is a finite CW,

B is smooth ($1/2$ saturated).

- $D(B\text{-dgmod}) \simeq D_{loc}(X, k) \subseteq D(X, k)$

$\quad\quad\quad\quad\quad\quad \shortparallel$

$\{E \mid \underline{H}^i(E) \text{ is locally constant } \forall i\}$

$\overline{D(X) \simeq D(B)}$. Many interesting invariants of X can be recovered from B.

ex. - X sm. proper scheme/k., char $k = 0$

$HH_*(B/k) \simeq \bigoplus H^*(X, \Omega^*_{X/k})$ (hodge cohomology)

- X finite CW cplx,

$HH_*(B/k) \simeq H_*(\mathcal{L}X)$, $\mathcal{L}X = Map(S^1, X)$.

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Part I: Finiteness results about dg algebras.

k a comm. ring (not necessarily char 0), everything over k.

Def: * B is a dg alg. $/k$. (associative with unit).

$D(B)$ = derived cat. of B-dg modules.

* B is proper if B is a perfect complex of k-modules.

(i.e. $B \underset{q.iso.}{\sim}$ bounded complex of proj. k-mod of finite type).

$\Leftrightarrow$ B is compact in $D(k)$.

Rmk: k field, B proper $\Leftrightarrow$ $H^\bullet(B)$ is finite dimensional. (need to derive when k is not a field).

* B is smooth if B is compact in $D(B \overset{(\mathbb{L})}{\otimes_k} B^{op})$.

(i.e. $[B, \oplus \bullet] \simeq \oplus [B, -]$).

("B. is of fin. homol. dim").

* B is saturated if it is smooth + proper.

* B is of finite type $\forall$ filtered system of dg-algebras B_α, we have:

$$[B, \operatorname{colim}_\alpha B_\alpha] \simeq \operatorname{colim}_\alpha [B, B_\alpha]$$

(Rmk: system is filtered, so hocolim = colim. actually this statement is w/ hocolim).

where $[B, C]$ = set of morphisms in $Ho(dg\text{-}alg/k)$

$$Ho(dg\text{-}alg./k) := (\text{quasi-isom})^{-1}(dg\text{-}alg/k).$$

(htpy analogue of being finitely presented as an algebra).

Prop: - finite type $\Rightarrow$ smooth

- saturated $\Rightarrow$ finite type.

(examples of dgas that are smooth but not finite type exist! e.g. $\mathbb{A}^1$ - infinite pts.?)

(⇒ q-cpct), (separated)

<u>Examples:</u> (*) X scheme, flat of fin. type /k. Then:

$$D_{qcoh}(X) \simeq D(B).$$

- B proper ⟺ X is proper.
- B smooth/k ⟺ X smooth /k.
- B finite type ⟺ X smooth /k.

(**) X finite CW, $D_{loc}(X, k) \simeq D(B)$

$B = C_*(\Omega X)$.

B is of finite type (thus smooth). Almost never proper.

(***) Q a finite quiver.

B = A(Q, k) (path algebra) of finite type /k.

<u>Rmk:</u> We will see that these notions (sm, proper, f. type) all have a "nice" categorical interpretation in terms of duality in a certain ⊗-2 cat of dg-cats. (saturated = most dualizable, the others just being 1/2 dualizable).

<u>Main theorems:</u>

(1) If B is a finite type dg-algebra /k, then there exists a algebraic moduli space of finite-dimensional B-dg modules

(2) There is a moduli space for saturated dg-algebras (up to quasi-isom.) (not up to Morita equiv, which would be the right notion) (false up to Morita equiv.?: only formal stack for Morita, can't be alg.)

<u>Rmk:</u> These moduli spaces are ?

- They should be, at least, algebraic stacks, because objects have automorphisms.
- In both cases, there are "higher automorphism groups":

$$
E \text{ is a dg-mod}/B \quad \begin{cases} \to \mathrm{aut}(E) \\ \to \mathrm{Ext}^{-1}(E,E) \leftarrow \text{2-automorphism gp. "self-htopies of } id_E\text{".} \\ \to \mathrm{Ext}^{1-i}(E,E) \leftarrow \text{i-automorphism group "self-htopies of } id_{id_{id_{\cdots E}}}\text{".} \end{cases}
$$

$E \in D(B)$

Conclusion: Moduli spaces in (1) of thm. <u>must be</u> algebraic ∞-stacks. (In the same way that alg 1-stacks have underlying schemes up to a stratification, these ∞-stacks do too).

<u>Case of dg algebras</u>: $B \in \mathrm{Ho}(\text{dg-alg}/k)$.

- $\mathrm{aut}(B)$ aut. group of B.
- $\mathrm{Der}^{-1}(B,B)$, where $\mathrm{Der}^{i}(B,M) = \mathrm{Ext}_{D(B^{op}\otimes B)}(I_B, M)$, $M \in D(B^{op}\otimes B)$

(close to HH^*, differs by a factor which is like $H(B)[\pm 1]$?)

- $\mathrm{Der}^{1-i}(B,B)$.

$0 \to I_B \to B^{op}\otimes B \xrightarrow{\text{mult.}} B \to 0$

Der, (not HH^*, b/c B is a rep't generator of a dg-cat, not a dg-cat. i.e. have a pointed dg-cat).

<u>Consequences</u>:

(*) Up to stratification, we get schemes of finite type $/k$ classifying saturated dg-algebras and fin.-dim'l dg-modules over a fin. type dg-alg, (+ $\mathrm{Aut}(E)$, $\mathrm{Ext}^{1-i}(E,E)$, ... are flat group schemes locally defined on those schemes).

(**) B fin. type, then $\exists$ an alg. space, loc. fin. type$/k$ M_B, classifying <u>simple objects</u>, i.e. $M_B \longleftrightarrow \left\{ E \in D(B) \;\middle|\; \begin{array}{l} \cdot \text{ fin. dim'l} \\ \cdot\ \mathrm{Ext}^{-i}(E,E) = 0 \\ \mathrm{Ext}^{0}(E,E) \simeq k \end{array} \right\}$

M_B comes equipped w/ a natural class $\alpha \in H^2_{et}(M_B, \mathbb{G}_m)$, obstruction for existence of a universal simple dg-module on M_B.

[$\overset{\alpha}{\text{torsion}} \longleftrightarrow \exists$ an "Azumaya alg." somewhere].

Another Application: X is a compact complex manifold.

If $D^b_{coh}(X) \simeq D_{perf}(B)$, B saturated, then X is algebraic.

Rmks on how to do this: write X as sub-cplx. manifold of said moduli space associated to B, do some more work.