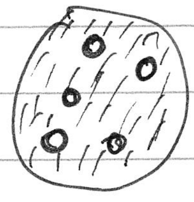
D. Kaledin — ~~Hochschild~~ 2-algebras as factorization algebras

Deligne conjecture: what is the natural structure on $CH^{\bullet}(A)$, A assoc. alg$/k$, char $k = 0$.

HH' complex

Operad of small discs:



D — n smaller discs inside.

O_n — configuration space.

operad of topological spaces

$H_{\bullet}(O_n, k)$ — Gerstenhaber operad.

$C_{\bullet}(O_n, k)$ DG operad, algebras over this operad are "2-algebras."

Thm: $CH_{\bullet}(A)$ is a 2-algebra up to quasi-isomorphism.

(so far, the way of doing this seems contrived/un-natural. e.g. $C_{\bullet}(O_n, k)$ is huge, need to pass to a q-iso guy, that's also huge)

Another way:

Factorization algebras:

D — unit disc

S — finite set.

$$\begin{tikzcd} S \arrow[rr] \arrow[dr, two heads] & & D \\ & S' \arrow[ur, hook] & \end{tikzcd}$$

$D^S = \mathrm{Maps}(S, D)$.

D^S is stratified by f.

$D^S_0 \subset D^S$ the open subset of inj. maps.

$f : S \twoheadrightarrow S'$ induces

$i_f : D^{S'} \longrightarrow D^S$

$j_f : D^S_f \subset D^S$ — open subset, larger than D^S_0. — is the union of $i_g(D^{S_1}_0) \subset D^S$, where $S \xrightarrow{g} S_1 \longrightarrow S'$, with composite f.

<u>Def</u>: A factorization alg. is a collection of the following:

(1) $\forall\, S$ $\mathcal{E}_\bullet$ – complex of const. sheaves of k-v. spaces on D^S, locally const. on stratum.

(2) $i_f^! : \mathcal{E}_\bullet(S) \xrightarrow{\sim} \mathcal{E}_\bullet(S')$ quasi-iso. (need to fix a dg-enhancement for ! functor)

(3) $j_f^* \mathcal{E}_\bullet(S) \xrightarrow{\sim} \boxtimes_{s \in S'} \mathcal{E}_\bullet(f^{-1}(s))$

w/ obv. compat. conditions.

(for any $f : S \to S'$,

$S = \coprod_{s \in S'} f^{-1}(s)$,

$D^S = \prod_{s \in S'} D^{f^{-1}(s)}$)

<u>Example</u>: $|S| = 2$ $\quad f : S \to pt.$ $\quad A_\bullet := \mathcal{E}_\bullet(pt)$

$D^S = D^2$.

$D \subset D^2$ Diagonal

$U = D^2 \setminus D$ open stratum

$i_f^! \mathcal{E}_\bullet(S) \cong \mathcal{E}_\bullet(pt.)$

$\mathcal{E}_\bullet(S)|_U \cong \mathcal{E}_\bullet(pt)^{\boxtimes 2}$.

The gluing data:

$$\underbrace{i_f^! j_{f!}}_{\text{compute.}} \mathcal{E}_\bullet(pt)^{\boxtimes 2} \to \mathcal{E}_\bullet(pt).$$

$$i_f^! j_{f!} A^{\boxtimes 2} \cong C_\bullet(U, A_\bullet^{\boxtimes 2}) \to A.$$

$$\shortparallel$$

$U \sim S^1$, $\quad (H_\bullet(S^1, k) \otimes A^{\otimes 2})_{\mathbb{Z}/2\mathbb{Z}}$

((coinv. w.r.t. involution).

$D^S_\circ$

$\bullet$ $\underset{O_S}{S}$, so factoriz. alg. should give something like operad over small discs. Hasn't been done b/c of technical issues w/ model structures.

Conjecture: The category of factorization algebras "up to quasi-iso" is equivalent to the category of 2-algebras up to quasi-iso.
(problems seem purely technical).
Advantages: there are nice combinatorial approximations of fact. algs that are incompatible w/ operad formalism.

Observation: Factorization algebras are the same as chiral algebras (Beilinson-Drinfeld), except that
(1) Replace constructible sheaves w/ D-modules
(2) Replace complexes w/ objects, and q-isos with isos.
(the examples in chiral algs. don't give us examples here b/c D-modules are mostly too big, not holonomic

Combinatorial models:

1. X stratified, nicely. (open strata are $K(\pi, 1)$)
$D(\mathrm{Shv}_{constr}(X)) \cong D(\mathrm{Functors}(\pi_1(X), k\text{-Vect}))$
some slight cheating: take D so we can glue by $(\cdot)^!$.

Def: Stratified fund. ~~groupoid~~ category of X $\overline{\pi_1}(X)$:
. objects $x \in X$
morphisms: $\gamma: [0,1] \to X$ s.t.
$\forall$ $X_1 \subset X$ (stratum), $\gamma^{-1}(X) = [a, 1] \subset [0,1]$,
$0 \le a \le 1$.
i.e. once we enter a stratum can't leave it.
(take htpy classes)

For D^S, $\overline{\pi_1(D^S)}$ is easy to describe.

~~objects~~ $f : S \twoheadrightarrow S'$ correspond to objects

$Aut(Id) = \mathcal{B}_S$ pure braid group.

$Aut(f) = \prod_{s \in S'} \mathcal{B}_{f^{-1}(s)}$

- In the def'n of factorization algebra, one can replace D^S_0 with $\overline{\pi_1(D^S)}$ everywhere.

  $\mathcal{B} \longrightarrow C^{\cdot}(k\text{-Vect})$
  sym. monoidal

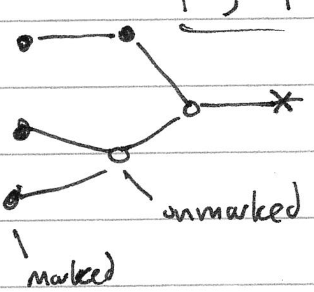
(Ref. Segal description of loop spaces: 1) $O_n^{E^\infty} \times X^n \longrightarrow X$

$$\begin{array}{ll} 2)\ X_n \longrightarrow X. & \\ \ \ \downarrow \wr & \uparrow \text{like 1),} \\ \ \ X^n & \text{w/o specifying } O_n^{E^\infty} \ldots \end{array}$$)

This is nice, but rep's of braid group are still hard.

- Second model: trees S (ref. Kont-Soibelman, others).

  T^S, $\overline{T^S}$ - Planar trees w/ a simple root, marked by S.



$S \hookrightarrow V(T)$ - the set of vertices.

Stable: unmarked vertices have valency ≥ 3.

trajectories really draw trees.

D^S_0 (almost) has a stratification with strata numbered by S-marked stable trees. (look at quad. diff'l of some special kinds) these trees are partially ordered by contraction w/ at least one unmarked pt.

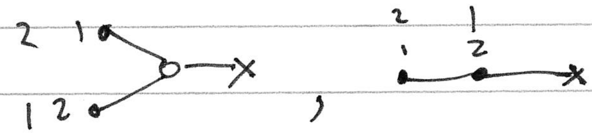
$\rightsquigarrow$ take nerve, geom. realization, htopy type is same as D_0'.

Nice b/c this acts on CH' essentially by definition?

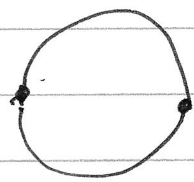
- these trees don't have an operad structure, & stratification incompatible?

~~[illegible]~~

For 2 points: 4 possibilities

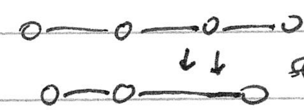


get a circle: 0 cells, 1-cells.



Instead of stable trees, we can consider all trees. They form a category T, (but inf. many possibilities, not a partial ordering)

Now:



several maps.

Functor $T \xrightarrow{V(-)} Set$.

$\tilde{T}$ the category of pairs (tree T, $v \in V(T)$)

$$\tilde{T}^S = \underbrace{\tilde{T} \times_T \tilde{T} \times_T \cdots \times_T \tilde{T}}_{S \text{ times}} \setminus diag$$

$\downarrow$

T

(T contractible, $\tilde{T}$ contractible, $\tilde{T}^{\times_T^S}$ contractible?, but $\tilde{T}^S \simeq T^S$.)

So, in some way, $\tilde{T}$ like a disc. Should be some gen. principle explaining this.

Q: Can one see more here, e.g. chiral algs instead of fact. algs.?