V. Ginzburg - Gerstenhaber - Batalin - Vilkovski structures on coisotropic intersections

joint w/ Baranovsky

X smooth alg$/\mathbb{C}$ $\supset Y, Z$ smooth subvarieties

sheaves (not using curly notation) $\cdot$ $\mathrm{Tor}(\mathcal{O}_Y, \mathcal{O}_Z)$ (sheaf of) graded comm. algs. $(\mathrm{Tor} = \mathrm{Tor}^{\mathcal{O}_X}, \text{etc.})$

$\mathrm{Ext}(\mathcal{O}_Y, \mathcal{O}_Z)$ gr. module over above

"Tor a $\mathcal{O}_{Y \cap Z}$"

From now, assume X alg. Poisson str., $\{\cdot, \cdot\}$, $P \in H^0(\Lambda^2 T_X)$

Y, Z are coisotropic, i.e. $\{J_Y, J_Y\} \subset J_Y$

$\iff P \longrightarrow 0$ in $\Lambda^2 N_{X/Y}$

Thm 1: (1) For any (smooth) coisotropic $Y, Z \subset X$,

$\mathrm{Tor}_\bullet(\mathcal{O}_Y, \mathcal{O}_Z)$ has a canonical Gerstenhaber algebra structure.

(2) Ext has a canonical Gerstenhaber mod. structure over Tor.

(To be honest, have no idea why this is true. Have a proof though...).

Often: Gerstenhaber algebras A are BV-algebras, i.e. there is a

$\Delta : A^\bullet \longrightarrow A^{\bullet - 1}$ • diff. operator of order ≤ 2.

• $\Delta^2 = 0$.

Given this, define

$$\{a, b\} := \Delta(ab) - \Delta(a)b - (-1)^{|a|} a\, \Delta(b)$$

($\Delta^2 = 0 \iff$ Jacobi).

Ex: (Y, P) Poisson mfold.

$X = Y \times Y \supset Y = Z$ diagonal. "$\Omega_Y^\bullet$"

$\mathrm{Tor}_\bullet(\mathcal{O}_Y, \mathcal{O}_Y) \simeq \Lambda^\bullet T_Y^*$. $\mathrm{Ext}^\bullet(\mathcal{O}_Y, \mathcal{O}_Y) \simeq \Lambda^\bullet T_Y$

$\Delta : \Omega^{\bullet} \to \Omega^{\bullet-1}$ $\quad \Delta = i_P d_{dR} + d_{dR} \cdot i_P$ "Lie deriv. w.r.t. P"

$\Delta^2 = 0 \Rightarrow$ Gerst. bracket on $\Omega_Y^{\bullet}$ "Koszul bracket."

Assume X is symplectic, Y, Z are Lagrangian.

fix a square root:

$\mathcal{L}_Y = K_Y^{1/2}$ half-forms, similar for Z.

Thm 2: $\exists$ canonical $\Delta : Ext^{\bullet}(\mathcal{L}_Y, \mathcal{L}_Z) \to Ext^{\bullet-1}(\mathcal{L}_Y, \mathcal{L}_Z)$.

Motivation for both theorems comes from 1) Behrend - Fantechi.

2) Kapustin - Rozansky; have the following conjecture:

$\exists$ triangulated category $\mathcal{C}(\mathcal{L}_Y, \mathcal{L}_Z)$ (lives "near" $Y \cap Z$) s.t.:

- $HH_{\bullet}(\mathcal{C}) = Ext(\mathcal{L}_Y, \mathcal{L}_Z)$
- $HH^{\bullet}(\mathcal{C}) = Tor(\mathcal{O}_Y, \mathcal{O}_Z)$.

Connes diff. $\longleftrightarrow \Delta$

Gerst. bracket $\longleftrightarrow \{\cdot, \cdot\}$

Special case: $X = T^*Y \hookleftarrow Y$ = zero section.

$f \in \mathcal{O}(Y)$, $Z = Graph(df) \subset T^*Y$.

$Y \cap Z$ = critical locus of f.

(in this case $\mathcal{L}_Y, \mathcal{L}_Z$ are the same & cancel?).

Using Koszul resolution,

$$Tor(\mathcal{O}_Y, \mathcal{O}_Z) = \mathcal{H}^{\bullet}(\Lambda T_Y^*, \underbrace{\wedge df}_{\text{differential}})$$

$$Ext(\mathcal{O}_Y, \mathcal{O}_Z) = \mathcal{H}(\Lambda T_Y, \iota_{df})$$

$d_{dR} \circlearrowright \Lambda^{\bullet} T_Y^*$ anti-commutes with $\wedge df$, descends to cohomology

$\Rightarrow$ ~~induced~~ d_{dR} induces a BV-~~th~~ differential on Tor.

Behrend: $\exists$ natural constructible fxn. on $crit(f)$, e, called the <u>local Euler obstruction</u> (MacPherson, 1971).

F (take v.c. functor, & dimension of stalks).

$$(\Omega_Y, d_{dR} + df)$$

deRham cplx of "D-mod of vanishing cycles".

B-F generalize this by locally patching up these T^*X models & checking that it works. This ~~doesn't work~~ requires analytic setting, fails algebraically! (algebraically, a sympl. form isn't locally exact in any topology).

Kapustin–Rozanski define

$$\mathcal{C}(\mathcal{O}_Y, \mathcal{O}_{Graph(df)}) = \text{Category of matrix factorizations ~~work~~ for } f.$$

i.e. object is a pair.

$$(E^+ \underset{\partial^-}{\overset{\partial^+}{\rightleftarrows}} E^-) \text{ vector bundles on } Y$$

$$\partial^+\partial^- = f \cdot Id$$
$$\partial^-\partial^+ = f \cdot Id$$

If f has an isolated singularity,

$$\mathcal{H}^\cdot(\Lambda T_Y, \iota_{df}) = \mathrm{Coker}[T_Y \xrightarrow{\iota_{df}} \mathcal{O}_Y] = \mathrm{Jacobi}(f)$$

(Rmk: in general, std. tricks to put a non-$\mathbb{Z}/2$ grading on MF).

Problem: MF isn't apriori local to $Y \cap Z$ (although true by Orlov)

$\mathcal{O}_X^\varepsilon$ nc deformation of $\mathcal{O}_X$ over $\mathbb{C}[\varepsilon]/(\varepsilon^2)$

as sheaf $/\!/$ $\mathcal{O}_X + \varepsilon\mathcal{O}_X$ e.g. $f * g = f \cdot g + \frac{\varepsilon}{2}\{f, g\}$.

$Y, Z \subset X$ coisotropic.

Let $\mathcal{L}$ be a line bundle on Y, $\rightsquigarrow \mathcal{L}^{\varepsilon}$ ← left modules over $\mathcal{O}_X^{\varepsilon}$.

$\mathcal{M}$ —— " —— Z. $\rightsquigarrow \mathcal{M}^{\varepsilon}$ ←

Idea: BV diff on $\mathcal{Z}$ comes from Ext 1st order deform of $(\mathcal{O}_X, \mathcal{L}, \mathcal{M})$, expanded in power series, ε term.

$\mathcal{E}xt_{\mathcal{O}_X}(\mathcal{L}, \mathcal{M})$

$\mathrm{Tor}^{\mathcal{O}^{\varepsilon}}(\mathcal{L}^{\varepsilon}, \mathcal{M}^{\varepsilon})$

(only resolution keeping track of product is bar resolution w/ shuffle prod. on affine subsets. don't know how to do globally)

Main problem: No preferred quantization of coisotropics (can take ε, 3 for $\mathcal{O}_X$). but magically, although Δ varies, the bracket doesn't change! (computation)

No canonical Δ.

Let Y be a Lagrangian submanifold in a symplectic manifold X, $\mathcal{L}_Y = K_Y^{1/2}$.

Additional data:

p splitting (required to be Lagrangian)

$$0 \to N^*_{X/Y} \to T^*_X|_Y \to T^*_Y \to 0$$

conormal bundle to Y.

$$f \in \mathcal{O}_X \rightsquigarrow df \rightsquigarrow p(df) \overset{\text{sympl. form}}{\rightsquigarrow} \xi_{p(df)}$$

← Ham. v.f. tangent to Y.

$\rightsquigarrow$ quantization of $\mathcal{L}_Y$:

$$f * l = f \cdot l + \frac{\varepsilon}{2} L_{\xi_{p(df)}}(l).$$

(Rmk: quantization of modules, unlike algebras, have obstructions)

Q: Let $Y \subset X$ a coisotropic subman.

$\mathcal{L}$ a line bdle on Y. Obstruction to quantization:

makes sense b/c P vanishes on $N \wedge N$.

$$P \in H^0(\Lambda^2 T_X) \longrightarrow \bar{P} \in H^0(Y, N_Y \otimes T_Y)$$

Poisson bivector.

Also have $\kappa \in H^1(X, T_X)$ non-splitness of $\mathcal{O}_X \to \mathcal{O}_X^\varepsilon \to \mathcal{O}_X$.

and $\alpha(N_{X/Y})$ "Atiyah class of $N_{X/Y}$" $\in H^1(Y, (\mathcal{E}nd\, N) \otimes \Omega_Y^1)$.

Prop: $\mathcal{L}$ deforms to a left $\mathcal{O}_X^\varepsilon$-module iff:

$$P \cdot v [2\, \mathrm{Id}_N \otimes c_1(\mathcal{L}) - \alpha(N)] + \bar{\kappa} = 0.$$

$$H^0(Y, N \otimes T_Y) \otimes \mathrm{Ext}^1(N \otimes T_Y, N) \longrightarrow H^1(Y, N)$$

- Symplectic case: $\bar{\kappa}$ vanishes, etc., & eqn becomes:

$$2c_1(\mathcal{L}) = c_1(K_X).$$

Rmk: There's a nice interpretation of obstructions to 2^{nd} order, but we have no idea about any others orders.