

Toen II

dg-categories

Part II: Local to global for dg-algebras

matrix factorizations

- examples:
- MF (X, f) , $f: X \rightarrow k$
 X non-affine.
 - $\mathcal{C}(\mathcal{L}_Y, \mathcal{L}_X)$ defined analytic locally.
 - X scheme, $\alpha \in H^2_{\text{et}}(X, \mathbb{G}_m)$

$D_\alpha(X) =$ derived cat. of α -twisted sheaves on X .

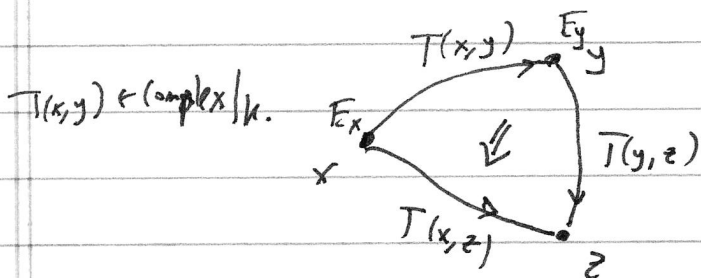
the gluing machinery is somehow final.

But: "Compact generators are as necessary to this subject as air to breathe"
 (Thomason)

non-final part in this talk: gluing of compact generators

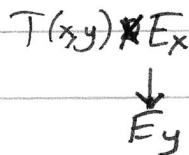
locally-presentable dg categories

dg cat: a dg algebra with many objects.



$T(x,y) \times T(y,z) \rightarrow T(x,z)$

A T -dg module is a dg functor $T \xrightarrow{E} \text{complexes}/k$



$\hookrightarrow D(T) = (\text{quasi-isom})^{-1} (T\text{-dg mod})$

A dg-functor $f: T \rightarrow T'$ is a quasi-equiv. if

- (i) $f_{x,y}: T(x,y) \rightarrow T'(f_x, f_y)$ is a quasi-isom $\forall x,y$
- (ii) $H^0(f): H^0(T) \rightarrow H^0(T')$ essentially surjective.

T dg cat. T -dg Mod is naturally a dg cat

$\hat{T} =$ full sub dg cat of T -dg mod consisting of
cofibrant dg modules
 (no quasi-free)

(link: cofibrant means H-projective).

$$D(T^{op}) \simeq H^0(\hat{T})$$

Def: T a dg cat (T is large)

(1) T is strictly locally presentable if $\exists T_0$ a small dg cat and S a small set of objects in \hat{T}_0 such that

$$T = \{E \in \hat{T}_0 \mid \underline{\text{Hom}}(K, E) \sim 0, \forall K \in S\}$$

(2) T is locally presentable if it is quasi-equiv. to a strictly loc. presentable dg cat.

(“very large” cat.)

Locally presentable dg cat/ k form a category $\mathcal{D}_g^{lp}(k)$, where morphisms are dg functors continuous (commute with \otimes) up to quasi-isomorphism.

$$(\mathcal{D}_g^{lp}(k) \subseteq \text{Ho}(\text{dgcat}_k))$$

↑
nonfull

Remark: $\mathcal{D}_g^{lp}(k)$ can be (should be) enhanced into a $(2, \infty)$ -category.
 (Some statements really need this enhancement to make sense)

$\mathcal{D}_g^{lp}(k)$ is a \otimes -sym. cat.

$$\begin{array}{ccc} T \longleftrightarrow (T_0, S) & T \otimes^{\text{ct}} T' \longleftrightarrow (T_0 \otimes T_0', S \boxtimes T_0' \vee T_0 \boxtimes S') \\ T' \longleftrightarrow (T_0', S') & \uparrow \\ & \text{continuous tensor product} \end{array}$$

Remark: continuous dg-functor $T \otimes^{\text{ct}} T' \rightarrow T''$

$$\begin{array}{c} \uparrow \text{1-1} \\ \text{dg-functors } T \otimes T' \rightarrow T'' \text{ continuous in each variable.} \end{array}$$

Prop.: (1) If $T \simeq \widehat{T}_0$ then T has duals in $\mathcal{D}_g^{lp}(k)$

$$\left[\exists T^v, \mathbb{1} \xrightarrow{id} T \overset{ct}{\otimes} T^v \xrightarrow{ev} \mathbb{1} \right]$$

\hat{k} " s.t. triangular identities]

(the converse seems true?)

$$T^v = (\widehat{T_0^{op}})$$

(2) B a dg-algebra is smooth \iff

$\widehat{B} \overset{ct}{\otimes} \widehat{B}^{op} \xleftarrow{id} \mathbb{1}$ is a compact morphism (i.e. has a right adjoint in $\mathcal{D}_g^{lp}(k)$). (Emb: always exist, but not necessarily continuous).

(Point: smooth, proper make sense in a 2-category, nothing to do w/ dg rats)

(3) B is proper iff $ev: \widehat{B} \overset{ct}{\otimes} \widehat{B}^{op} \rightarrow \mathbb{1}$ is compact.

(also means r.adj. has a r.adj. similar statements for tuples, etc.)

(4) B saturated $\iff \widehat{B}$ has a dual in $\mathcal{D}_g^{lp,c}(k) \subseteq \mathcal{D}_g^{lp}(k)$, the subcategory of compact dg-functors.

Emk: All of these notions make sense in many \otimes -2cat.

(Lurie: fully dualizable objects).

dg rat with descent:

Def: $k \rightarrow k'$ a faithfully flat morphism (of rings)

$$k \rightarrow k' \implies k' \otimes_k k' \cong \dots \otimes_k k' \quad \text{cosimplicial gadget}$$

$T \in \mathcal{D}_g^{lp}(k)$

$$T \rightarrow T \overset{ct}{\otimes} \widehat{k'} \cong T \overset{ct}{\otimes} (\widehat{k' \otimes_k k'}) \dots$$

$$\parallel$$

$$T_{k'/k} \quad \text{cosimplicial object}$$

T has descent if $T \rightarrow \lim T_{k'/k}$ is a quasi-equivalence

$\forall k \rightarrow k'$ + universally

(If T is compactly generated, ~~then~~ it has descent)

Prop: (1) \hat{T}_0 has descent

(2) locally presentable dg categories with descent are stable by limits.

$$\begin{array}{ccc}
 (3) \quad k \longrightarrow \mathcal{D}_g^{lp, desc}(k) & & \\
 \downarrow & \searrow \otimes_k^{ct} \uparrow & \\
 k' \longrightarrow \mathcal{D}_g^{lp, desc}(k') & & \text{(base change functor)}
 \end{array}$$

is a stack for the fpqc topology.

Def: X a scheme, a loc. presentable dg cat. $/X$ is:

$$\begin{array}{ccc}
 \forall \text{Spec } k \xrightarrow{u} X, T_u \in \mathcal{D}_g^{lp, desc}(k) & & \\
 \uparrow f & \nearrow v & \\
 \text{Spec } k' & & \phi_f: T_u \otimes_k^{ct} k' \xrightarrow{\sim} T_v
 \end{array}$$

+ ... coherence, i.e.

$$\mathcal{D}_g^{lp, desc}(X) := \lim_{\text{Spec } k \rightarrow X} \mathcal{D}_g^{lp, desc}(k)$$

$$T \in \mathcal{D}_g^{lp, desc}(X), \text{ then } \Gamma(X, T) := \lim_{\text{Spec } k \xrightarrow{u} X} T_u \in \mathcal{D}_g^{lp, desc}(\mathbb{Z})$$

base ring of X

Theorem: Suppose that X is quasi-compact and quasi-separated.

$T \in \mathcal{D}_g^{lp, desc}(X)$, if $\exists X' \rightarrow X$ fpqc covering s.t. $\Gamma(X', T|_{X'})$ has a compact generator, then $\Gamma(X, T)$ has a compact generator

Application: X scheme, qc, qs
 $\alpha \in H_{\text{et}}^2(X, \mathbb{G}_m)$

- $D_\alpha(X)$ has a compact generator E

$\mathcal{R}\text{End}(E) = \mathcal{A}$ sheaf of dg algebras $/X$

\mathcal{A} is Azumaya: $\mathcal{A} \otimes_{\mathcal{O}_X}^L \mathcal{A}^{\text{op}} \xrightarrow{\sim} \mathcal{R}\text{End}_{\mathcal{O}_X}(\mathcal{A})$

Cor: $H_{\text{et}}^2(X, \mathbb{G}_m) \times H_{\text{et}}^1(X, \mathbb{Z})$
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{dg Azumaya algebras $/X$ }
 Morita equiv.