Toen II

Part II: local to global for ~~dg-algebras~~ dg-categories

examples:
- $MF(X, f)$, $f : X \to k$ — matrix factorizations
  X non-affine.
- $\mathcal{C}(\mathcal{L}_Y, \mathcal{L}_X)$ defined analytic locally.
- X scheme, $\alpha \in H^2_{et}(X, \mathbb{G}_m)$

$D_\alpha(X)$ = derived cat. of α-twisted sheaves on X.

the gluing machinery is somehow formal.

But: " Compact generators are as necessary to this subject as air, to breathe "
(Thomason)

non-formal part in this talk: gluing of compact generators

Locally presentable dg categories

dgCat: a dg algebra with many objects.

$T(x,y) \in$ complex$/k$.

$$\begin{tikzcd} & y \ (E_y) \arrow[dd, "T(y,z)"] \\ x \ (E_x) \arrow[ur, bend left, "T(x,y)"] \arrow[dr, bend right, "T(x,z)"'] \arrow[r, phantom, "\Downarrow"] & {} \\ & z \end{tikzcd}$$

$T(x,y) \otimes T(y,z) \to T(x,z)$

A T-dg module is a dg functor $T \xrightarrow{E}$ Complexes$/k$

$$T(x,y) \otimes E_x \to E_y$$

$\rightsquigarrow D(T) = (\text{quasi-isom})^{-1}(T\text{-dg mod})$

A dg-functor $f : T \to T'$ is a quasi-equiv. if
(i) $f_{xy} : T(x,y) \to T'(fx, fy)$ is a quasi-isom $\forall x, y$
(ii) $H^0(f) : H^0(T) \to H^0(T')$ essentially surjective.

T dg cat. T-dg Mod is naturally a dg cat

$\hat{T}$ = full sub dg cat of T^{op}-dgmod consisting of
cofibrant dg modules
($\sim$ quasi-free)

(Rmk: cofibrant means H-projective).

$D(T^{op}) \simeq H^0(\hat{T})$

Def: T a dg cat (T is large)

(1) T is strictly locally presentable if $\exists$ T_0 a small dg cat and S a small set of objects in $\hat{T}_0$ such that
$T = \{E \in \hat{T}_0 \mid \underline{Hom}(K, E) \sim 0, \forall K \in S\}$

(2) T is locally presentable if it is quasi-equiv. to a strictly loc. presentable dg cat.

("very large" cat.)

Locally presentable dgcat/k form a category $\mathcal{D}_g^{lp}(k)$, where morphisms are dg functors continuous (commute with $\oplus$) up to quasi-isomorphism.

$(\mathcal{D}_g^{lp}(k) \underset{\text{non-full}}{\subseteq} Ho(dgcat_k))$

Rmk: $\mathcal{D}_g^{lp}(k)$ can be (should be) enhanced into a $(2,\infty)$-category.
(Some statements really need this enhancement to make sense).

$\mathcal{D}_g^{lp}(k)$ is a $\otimes$-sym. cat.

$T \longleftrightarrow (T_0, S)$

$T' \longleftrightarrow (T_0', S')$

$T \otimes^{ct} T' \longleftrightarrow (T_0 \otimes T_0', S \boxtimes T_0' \cup T_0 \boxtimes S')$

continuous tensor product

Rmk: continuous dg-functor $T \otimes^{ct} T' \to T''$

$\updownarrow$ 1-1

dg-functors $T \otimes T' \longrightarrow T''$ continuous in each variable.

Prop.: (1) If $T \simeq \hat{T}_0$ then T has duals in $\mathcal{D}_g^{lp}(k)$

$$\left[\exists\, T^\vee,\ \underset{\hat{k}}{\overset{\shortparallel}{\mathbb{1}}} \xrightarrow{id} T \hat{\otimes}^{ct} T^\vee \xrightarrow{ev} \mathbb{1} \quad \text{s.t. triangular identities.} \right]$$

(the converse seems true?)

$T^\vee = \widehat{(T_0^{op})}$

(2) B a dg-algebra is smooth $\Longleftrightarrow$

$\hat{B} \otimes^{ct} \widehat{B^{op}} \xleftarrow{id} \mathbb{1}$ is a compact morphism (i.e. has a right adjoint in $\mathcal{D}_g^{lp}(k)$). (Rmk: always exist, but not necessarily continuous).

(Point: notions of smooth, proper make sense in a 2-category, nothing to do w/ dg cats)

(also means r. adj. has a r. adj. Similar statements for triples, etc.).

(3) B is proper iff $ev: \hat{B} \otimes^{ct} \widehat{B^{op}} \longrightarrow \mathbb{1}$ is compact.

(4) B saturated $\Longleftrightarrow$ $\hat{B}$ has a dual in $\mathcal{D}_g^{lp,c}(k) \subseteq \mathcal{D}_g^{lp}(k)$, the subcategory of compact dg-functors.

Rmk: All of these notions make sense in many $\otimes$-2cat.

(Lurie: fully dualizable objects).

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dg cat with descent:

Def: $k \to k'$ a faithfully flat morphism. (of rings?)

$$k \to k' \rightrightarrows k' \otimes_k k' \ \overrightarrow{\underset{\to}{\to}}\ \cdots\ k'^{\otimes_k^{n+1}}$$ cosimplicial gadgets

$T \in \mathcal{D}_g^{lp}(k)$

$$T \to T \otimes^{ct} \hat{k'} \ \overrightarrow{\underset{\to}{\to}}\ T \otimes^{ct} \widehat{(k' \otimes_k k')} \ - -$$

$$\shortparallel$$

$T^{\bullet}_{k'/k}$ cosimplicial object

T has descent if $T \longrightarrow \lim T^{\bullet}_{k'/k}$ is a quasi-equivalence

$\forall\ k \to k'$ + universally

(If T is compactly generated, ~~then~~ it has descent)

Prop: (1) $\hat{T}_0$ has descent

(2) Locally presentable dg categories with descent are stable by limits.

(3)
$$\begin{tikzcd} k \arrow[r, mapsto] \arrow[d] & \mathcal{D}_g^{lp,desc}(k) \arrow[d, "\hat{\otimes}^{ct} \hat{k'}"] \\ k' \arrow[r, mapsto] & \mathcal{D}_g^{lp,desc}(k') \end{tikzcd}$$
(base change functor) is a stack for the fpqc topology.

Def: X a scheme, a loc. presentable dg cat. $/X$ is:

$$\begin{tikzcd} \forall\ \operatorname{Spec} k \arrow[r, "u"] & X \\ \operatorname{Spec} k' \arrow[u, "f"] \arrow[ur, "v"'] \end{tikzcd}, \quad T_u \in \mathcal{D}_g^{lp,desc}(k), \quad \phi_f : T_u \otimes_k^{ct} \hat{k'} \xrightarrow{\sim} T_v$$

$+ \cdots$ coherences, i.e.

$$\mathcal{D}_g^{lp,desc}(X) := \lim_{\operatorname{Spec} k \to X} \mathcal{D}_g^{lp,desc}(k)$$

$T \in \mathcal{D}_g^{lp,desc}(X)$, then

$$\Gamma(X,T) := \lim_{\operatorname{Spec} k \xrightarrow{u} X} T_u \in \mathcal{D}_g^{lp,desc}(\mathbb{Z})$$
base ring of X

Theorem: Suppose that X is quasi-compact and quasi-separated. $T \in \mathcal{D}_g^{lp,desc}(X)$, if $\exists\ X' \to X$ fppf covering s.t. $\Gamma(X', T_{|X'})$ has a compact generator, then $\Gamma(X,T)$ has a compact generator.

Application: X scheme, qc, qs

$\alpha \in H^2_{et}(X, \mathbb{G}_m)$

- $D_\alpha(X)$ has a compact generator $\mathcal{E}$

$\mathbb{R}\mathrm{End}(\mathcal{E}) = \mathcal{A}$ sheaf of dg algebras $/X$

$\mathcal{A}$ is Azumaya: $\mathcal{A} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{A}^{op} \xrightarrow{\sim} \mathbb{R}\underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{A})$.

Cor: $H^2_{et}(X, \mathbb{G}_m) \times H^1_{et}(X, \mathbb{Z})$
$$\wr\!\!\mid$$
$\{\text{dg Azumaya algebras}/X\}/\text{Morita equiv.}$