Block – TBA

Antecedents

Using deformation quantization to get Fukaya:

1) Bressler, Soibelman
2) Nast, Tsygan
3) Tsygan
4) Kapustin, A-branes and nc geometry
5) Kapustin – Witten
6) Polesello, Schapira (Kashiwara)
7) Tamarkin

pre-Fukaya

8) Guillemin – Sternberg (associate certain Fourier – Integral (calculus) to Lagrangians in products)
9) Block, Getzler, Quantization of foliations.

Framework: $(A^\bullet, d, c)$ a curved dga.

1) $d(ab) = da\, b + (-1)^a a\, db$ $\qquad A^k = 0 \quad k < 0.$
2) $d^2 = [c, \cdot]$, $\quad c \in A^2$
3) $dc = 0$

Let $E^\bullet$ be a graded f.g. projective module over $A = A^0$.

~~Connection~~ $\mathbb{Z}$-graded connection: $\mathbb{E} : E^\bullet \longrightarrow (E^\bullet \otimes_A A^\bullet)$

$$\mathbb{E}(ea) = \mathbb{E}(e)\, a + (-1)^e e\, da.$$

$\mathbb{E}$ is of global degree 1.

$$\mathbb{E} = \mathbb{E}^0 + \mathbb{E}^1 + \mathbb{E}^2 + \cdots$$

$$\mathbb{E}^k : E^\bullet \longrightarrow E^{\bullet - k + 1} \otimes_A A^k.$$

We say $(E, \mathbb{E})$ is cohesive if $\mathbb{E} \circ \mathbb{E}(e) = -e \cdot c$.

$P_{\mathcal{A}}$ denotes the dg category of cohesive modules $\quad \mathcal{A} = (A^\bullet, d, c)$.

On X a complex manifold, $\qquad \mathcal{A} = (A^{0,\bullet}, \bar\partial, 0)$.

$E^\bullet$ = graded vector bundle,

$$\mathbb{E}^0 : E^\bullet \to E^{\bullet+1}$$
$$\mathbb{E}^1 : E^\bullet \to E^\bullet \otimes_A A^{0,1}$$
$$\mathbb{E}^0\mathbb{E}^1 + \mathbb{E}^1\mathbb{E}^0 = 0$$
$$\mathbb{E}^0\mathbb{E}^0 = 0, \quad \mathbb{E}^1 \cdot \mathbb{E}^1 = 0$$

A complex of hol. vector bundles.

For any coherent sheaf on X, you have a resolution by a cohesive module.

$$\mathbb{E}^0 + \mathbb{E}^1 + \cdots$$
$$\mathbb{E}^0 \cdot \mathbb{E}^0 = 0, \quad \mathbb{E}^1 \cdot \mathbb{E}^0 + \mathbb{E}^0 \cdot \mathbb{E}^1 = 0.$$
$$\mathbb{E}^0\mathbb{E}^2 + \mathbb{E}^1\mathbb{E}^1 + \mathbb{E}^2\mathbb{E}^0 = 0 \quad \text{(homotopic to 0)}$$

! coherences.

(don't always have one by locally frees here).

Theorem: $P_{\mathcal{A}}$ is dg equivalent to the dg category of ~~complexes~~ perfect complexes on X.

(Rmk: this category often won't be saturated!)

Suppose we have X a complex manifold,

$$\alpha \in H^2(X; \mathcal{O}^\times) \longrightarrow \beta \in H^3(X; \mathbb{Z}) \qquad \alpha \text{ goes to } 0$$
$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{e^{2\pi i}} \mathcal{O}^\times \to 0.$$

then α pulls back to $H^2(X, \mathcal{O})$ and can be represented by

$$B \in A^{0,2}(X), \quad \bar\partial B = 0$$

Theorem: $\mathcal{A} = (A^{0,\bullet} X, \bar\partial, B)$,

$P_{\mathcal{A}}$ is dg-equivalent to twisted sheaves on (X, α).

Let V be a real vector space, $\Lambda \subseteq V$ a lattice, J a complex structure on V.

$X = V/\Lambda$ is a complex torus.

$X^\vee = \bar{V}^\vee / \Lambda^\vee = \mathrm{Pic}^0(X)$.

Schwartz sequences (equiv. by Fourier transform)

$$C^\infty(X^\vee) = \mathcal{S}(\Lambda) = A = \{ \textstyle\sum a_\lambda \lambda \mid \sum_\lambda (1+|a_\lambda|)^k < \infty \}.$$

Take $B \in A^2(X)$, $dB = 0$,

$$B = B^{1,0} + B^{1,1} + B^{0,2}$$

Form $\mathcal{A} = (A^{0,0}X, \bar\partial, B^{0,2})$ ⇜ (deformed in gerbe direction)

Kapustin-Orlov showed in a physics calculation that dual torus becomes: $(\mathcal{B}^{\cdot}_{B}, \bar\partial, 0) \to \mathcal{B}$.

$B \in A^2(X)$. $B \in \Lambda^2 V^*$, $dB = 0$,

Define $\sigma : V \times V \to U(1)$

$$\sigma(v_1, v_2) = e^{2\pi i B(v_1, v_2)}$$

$\Lambda \subseteq V$, $\mathcal{S}\Lambda$, $[\lambda_1][\lambda_2] = \sigma(\lambda_1, \lambda_2)[\lambda_1 + \lambda_2]$

$$\mathcal{B} = \mathcal{S}^*(\Lambda, \sigma),$$

Set $\mathcal{B}^{\cdot} = \mathcal{B} \otimes \Lambda^{\cdot} V_{1,0}$, $\bar\partial \lambda = 2\pi i \lambda\, D^{\cdot}(\lambda)$, $D : V \to V_{1,0}$.

If $B = 0$, $\mathcal{B}^{\cdot} \equiv (A^{0,\cdot}(X^\vee), \bar\partial)$.

In general, Perfect category of a complex m'fold is schwach.

In the definition of cohesive module, just weaken the condition that $E^{\cdot}$ is finitely generated.

(but keep projectivity).

Assume: $(\mathcal{A}^{\cdot}, d, c)$ topological algebra (nuclear Frechet)

$E^{\cdot}$ is projective in the sense of being direct summands of $A \hat\otimes V$, V = nuclear Frechet vector space.

Such objects form $q\mathcal{P}_{\mathcal{A}}$.

An object $(M, \mathbb{M})$ is $q\mathcal{P}_{\mathcal{A}}$ gives a module over $\mathcal{P}_{\mathcal{A}}$, call it $\hat h_M$ (not exactly Yoneda).

Theorem: An object $(M, \mathbb{M})$ is $q\mathcal{P}_{\mathcal{A}}$ is quasi-representable (i.e. M is quasi-isomorphic to h_E for $E \in \mathcal{P}_{\mathcal{A}}$) if ~~$\mathbb{M}^0$ is A-nuclear~~. (A-nuclear means approximable by finite-rank operators) $\exists\, k, T$ s.t. $\mathbb{M}^0 k + k\mathbb{M}^0 = 1 - T$, T is A-nuclear.

Def: $T: M \to M$ is A-nuclear if $\exists\, \lambda_i \in \mathbb{C}$, $\sum_i |\lambda_i| < \infty$.

$m_i \in \mathrm{Hom}_A(M, A)$, $n_i \in M$, both bounded.

$T(m) = \sum_i \lambda_i \cdot n_i \cdot m_i(m)$

(Grauert's direct image theorem)

Corollary: If $f: X \to Y$ is a proper morphism of complex manifolds, then

$f_*: Ho\, P_X \to Ho\, P_Y$.

Example: If $U \subseteq X$,

$(M, \bar\partial) = (A^{0,\bullet}(U), \bar\partial)$

$A^{0,\bullet}(U)$ is projective over $A\dot{X}$, not finitely generated.

Generalized complex manifold X

$\mathbb{J}: (TX \oplus T^\vee X) \circlearrowleft$

$\mathbb{J}^2 = -1$, + integrability condition.

$\mathbb{J} = \begin{pmatrix} I & P \\ B & J \end{pmatrix}$

Special cases: 1) Complex manifold $\begin{pmatrix} J & 0 \\ 0 & J^\vee \end{pmatrix}$

Holomorphic Poisson manifolds $\begin{pmatrix} J & P \\ 0 & J^\vee \end{pmatrix}$

Holomorphic pre-symplectic manifold $\begin{pmatrix} J & 0 \\ B & J^\vee \end{pmatrix}$

Symplectic : $\begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}$

Very coarsely: $\begin{pmatrix} I & P \\ B & J \end{pmatrix}$ $I, J \longleftrightarrow$ complex str.

P Poisson

B is gerbe (symplectic).

Kapustin If (X, ω) is a symplectic manifold with a big brane;
$\mathcal{L} \to X$ a line bundle
∇, $F_\nabla \omega^{-1} F_\nabla = -\omega$.

Define $J = F\omega^{-1}$, then by above $J^2 = -1$, it happens to be automatically integrable. A-branes on $(X, \omega) \longleftrightarrow$
B-branes on a n.c. deformation of X to a holomorphic Poisson manifold.

Can verify that P of the n.c. manifold is equivalent P_{mirror}.

Q: (Toën): Is there any reason not to work w/ algebraic spaces here in addition to manifolds? A: No.