Lunts – Uniqueness of Enhancement for triangulated categories I.

(w/ D. Orlov)

- K – commutative ring. Everything is K-linear. $\otimes = \otimes_K$
- Δ-ted cats (Grothendieck, Verdier) ~ 50 years ago

Big problem: Cone is not functorial

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] & C(f) \arrow[d, dashed, "\text{not unique.}"] \\ X' \arrow[r, "f'"] & Y' \arrow[r] & C(f') \end{tikzcd}$$

Problems: • T triangulated category

$$T^* := \mathrm{Fun}_{\mathrm{cohom}}(T, Ab) \qquad \text{(cohomological functors)}$$

we know this should be Δted but can't prove it.

• T, T' Δted, want tensor product $T \otimes T'$ – Δ^{ted} cat.

want: $D(A) \quad D(B) \Longrightarrow D(A \otimes B)$, or more generally

X, Y scheme. $\quad D(X) \quad D(Y) \qquad D(X \times Y)$

can't do this with just Δ^{ted} category structure.

Solution: Often, for $X, Y \in T$,

$$\mathrm{Hom}_T(X, Y) \text{ is naturally } = H^0(\mathcal{H}om(X, Y)).$$

Ex: $T = D(A)$. $\quad X, Y$ – A-modules.

$P^\bullet \longrightarrow X$ proj. resolution.

$$\mathrm{Hom}(X, Y[n]) = \mathrm{Ext}^n_A(M, N) = H^0(\mathrm{Hom}_A(P^\bullet, Y[n]))$$

~~Def~~ – A DG category can be pre-triangulated; if it is, the cone is functorial (!).

- If $\mathcal{A}$ is a DG category,
$Ho(\mathcal{A})$ - usual category.
$\shortparallel$
Toen's $H^0(A)$

If $\mathcal{A}$ is pre-triangulated, then $H_0(A)$ is Δ-ted.

<u>Def</u>: An enhancement of a Δ-ted cat. T is a pair $(\mathcal{C}, G)$ such that

- $\mathcal{C}$ pre-triangulated
- $G : T \xrightarrow{\sim} H_0(\mathcal{C})$

<u>Existence of enhancement</u>: (Keller) it exists for all <u>algebraic</u> cat. := stable cat. of a Frobenius exact category

<u>Example</u>

1) A - small DG cat. $T = \underline{D(\mathcal{A})}$ - Δ-ted cat.
$$\mathcal{P}(A) = \{ P \in D(A) \mid Hom(P, E) \text{ acyclic if } E \text{ is acyclic} \}$$
$$Ho(\mathcal{P}(A)) \simeq D(A)$$
2) $\mathcal{E}$ - Groth. ab. category
$$D(\mathcal{E}) = Ho(I(\mathcal{E})), \quad I(\mathcal{E}) \text{ - h-injective complexes}$$
(point: there are enough of these).

<u>Uniqueness</u>:

<u>Def</u>: A DG Functor $F : \mathcal{C} \to \mathcal{D}$ is a quasi-equiv. if $Ho(F) : Ho(\mathcal{C}) \to H_0(\mathcal{D})$ is essentially surjective &
$$F : Hom^{\bullet}_{\mathcal{C}}(X, Y) \xrightarrow[q\text{-iso}]{} Hom_{\mathcal{D}}(F(X), F(Y))$$

<u>Def</u>: T has a unique enhancement if it has one and given two $(\mathcal{C}, G), (\mathcal{C}', G')$ then $\mathcal{C} \simeq \mathcal{C}'$.

T has a ~~strongly~~ strongly unique enhancement if in addition

$$\begin{tikzcd} & T \arrow[dl, "G"'] \arrow[d, "G'"] & \\ Ho(\mathcal{C}) \arrow[r, "\exists"'] & Ho(\mathcal{C}') \end{tikzcd}$$

$\uparrow$ comes from a q. eq. $\mathcal{C} \sim \mathcal{C}'$

Examples of uniqueness

A - algebra $D(A)$

Suppose $\mathcal{C}$ is a pre-triang. cat. & $G: Ho(P(A)) \xrightarrow{\sim} Ho(\mathcal{C})$.

Let $\mathcal{E} = Hom_{\mathcal{C}}(G(A), G(A))$ - DG-alg.

$F: \mathcal{C} - \mathcal{E}$-mod

$X \mapsto Hom_{\mathcal{C}}(G(A), X)$

$$Ho(F): Ho(\mathcal{C}) \xrightarrow{\sim} D(\mathcal{E})$$

So it suffices to compare $P(A)$ & $P(\mathcal{E})$

$$A \leftarrow \tau_{\le 0}\mathcal{E} \xhookrightarrow{i} \mathcal{E}$$

$$P(A) \xleftarrow[\sim]{p^*} P(\tau_{\le 0}\mathcal{E}) \xrightarrow[\sim]{i^*} P(\mathcal{E})$$

not strongly unique (almost never holds)

Counterexample to existence

(Muro, Schwede, Strickland)

$R = \mathbb{Z}/4$.

$F(R)$ - finitely gen. free R-modules

Thm: $F(R)$ has a Δ-structure s.t. 1) $[1] = id$.

2) $R \xrightarrow{2} R \xrightarrow{2} R \xrightarrow{2} R$ is an exact triangle.

$\Rightarrow$ no enhancement.

Counterexample to uniqueness

$k = \mathbb{F}_p$, $\Lambda_1 = \mathbb{Z}/p^2$, $\Lambda_2 = k[\varepsilon]/\varepsilon^2$.

$$C_1 \quad \xrightarrow{p} \Lambda_1 \xrightarrow{p} \Lambda_1 \xrightarrow{p} \cdots$$

$$C_2 \quad \xrightarrow{\varepsilon} \Lambda_2 \xrightarrow{\varepsilon} \Lambda_2 \to \cdots$$

$$\mathcal{E}_1 = \mathrm{Hom}(C_1, C_1), \quad \mathcal{E}_2 = \mathrm{Hom}(C_2, C_2)$$

Then $\mathrm{Ho}(\underline{P(\mathcal{E}_1)}) \overset{\Delta}{\simeq} \mathrm{Ho}(\underline{P(\mathcal{E}_2)})$, but

$P(\mathcal{E}_1)$ & $P(\mathcal{E}_2)$ are distinguished by K-theory.

Def: An object $Z \in T$ is compact if

$$\mathrm{Hom}(Z, \bigoplus_i Y_i) = \bigoplus_i \mathrm{Hom}(Z, Y_i)$$

Ex: X - q comp. sep.

$$D(\mathrm{QCoh}\, X)^c = \mathrm{Perf}.$$

Thm A: (k-field) Let $\mathcal{A}$ be a small category. Let $L \subset D(\mathcal{A})$ be a (k-linear, has sums, –). localizing subcat s.t. Yoneda

$$D(\mathcal{A}) \xrightarrow{\pi} D(\mathcal{A})/L \text{ has a right adjoint.}$$

Assume:

b) $\forall\, Y, Z \in \mathcal{A}$, $\mathrm{Hom}(\underline{\pi(Y)}, \underline{\pi(Z)[i]}) = 0$ for $i < 0$

a) $\forall\, Y, \cancel{} \in \mathcal{A}$, $\pi(Y) \in D(\mathcal{A})/L$ is compact.

Then $D(\mathcal{A})/L$ has a unique enhancement.

Thm B: In Thm A, assume that L is generated by $D(A)^c \cap L = L^c$ (cpct. objects). Then

$(D(A)/L)^c$ also has a unique enhancement.

small, doesn't have, e.g. sums

Plan of proof of Thm. A:

Def (Keller): Let $\mathcal{A}, \mathcal{B}$ DG categories.
A quasi-functor $\mathcal{A} \to \mathcal{B}$ is a DG functor
$F: \mathcal{A} \longrightarrow \mathcal{B}\text{-mod}$ s.t. $\forall\ X \in \mathcal{A}$, $F(X)$ is quasi-iso.
to a representable DG $\mathcal{B}$-module.

• A quasi-functor induces $H_0(F): H_0(\mathcal{A}) \to H_0(\mathcal{B})$.

(Toen): Consider the localization of $DGcat_k$ w.r.t. quasi-equivalences. Then morphisms in this localized cat. are in bijection with isom. classes of quasi-functors.

Thm A: Existence of enhancement for $D(A)/\mathcal{L}$
(Keller, Drinfeld).

Drinfeld:
$$\begin{array}{c} H_0(P(A)) = D(A) \\ \cup \\ H_0(\mathcal{L}) = L \end{array}$$

Drinfeld: $\exists$ a DG functor
$P(A) \longrightarrow P(A)/\mathcal{L}$ so that
$$H_0(P(A))/H_0(\mathcal{L}) \simeq H_0(P(A)/\mathcal{L})$$

Given a functor $F: P(A) \to \mathcal{D}$ s.t. $H_0(F)$ factors through $H_0(P(A)/\mathcal{L})$ then there exists a q-funcor $\tilde{F}: P(A)/\mathcal{L} \to \mathcal{D}$

s.t.
$$\begin{tikzcd} P(A) \arrow[r, "F"] \arrow[d] & \mathcal{D} \\ P(A)/\mathcal{L} \arrow[ur, dashed, "\tilde{F}"'] & \end{tikzcd}$$

Uniqueness: Assume $H_0(\mathcal{C}) \underset{G}{\simeq} H_0(P(A)/\mathcal{L})$

$$
\begin{tikzcd}
 & D(\mathcal{A}) \arrow[r, leftrightarrow, "\text{Yoneda}"] \arrow[d, "\pi"] & \mathcal{A} \arrow[dll, bend left, "F"] \\
H_0(\mathcal{C}) & D(\mathcal{A})/L \arrow[l, "G"', "\sim"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
P(\mathcal{A}) \arrow[d] & \\
P(\mathcal{A})/\mathcal{L} \arrow[r, dashed] \arrow[dr, dashed, "?"] & \mathcal{C}. \\
 & P(\mathcal{E})
\end{tikzcd}
$$

Let $\mathcal{E} \subset \mathcal{C}$ be a DG subcategory with objects $F(\mathcal{A})$

- From assumptions, we know $\mathcal{C} \xrightarrow[\text{q.e.}]{} P(\mathcal{E})$ (uses compact image assumption a?)

$$\mathcal{A} \xrightarrow{F} H_0(\mathcal{E}) \xleftarrow[\sim]{\text{q-equiv.}} \tau_{\leq 0}\, \mathcal{E} \longrightarrow \mathcal{E}$$

$\Rightarrow$ quasi-functor $P(\mathcal{A}) \to P(\mathcal{E})$.

Rmk 1: Probably can prove similar results for a general ring k.

Rmk. 2: Cannot apply to "periodic categories"

Problem: $T \xrightarrow{F} T'$

$$\mathcal{C} \overset{\exists F'}{\dashrightarrow} \mathcal{C}'$$