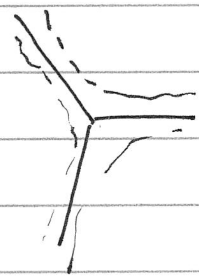
Kontsevich I – Singular Lagrangian Branes

Goal: Generalize Lag'n submanifolds the way of vector bundles ⇝ coh. sheaves

Singular Lagrangian:

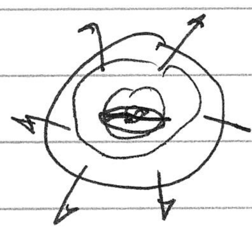
$L \subset (X, \omega)$ singular subset, should have "special singularities" – don't know good description



$U \supset L$ (spine of U) ε-nhood, exact symplectic manifold.

Right terminology for U is Liouville manifold

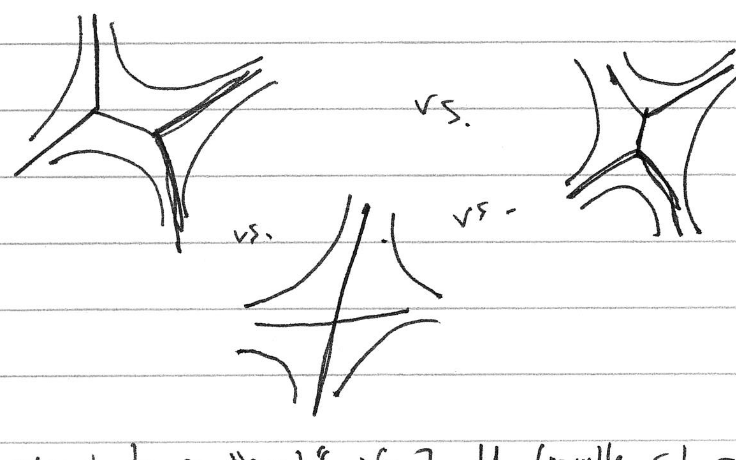
Def: Liouville mfold, (U, ω) exact sympl. mfold w/ boundary ∂U.
$\exists$ vector field ξ, s.t. $\mathrm{Lie}_\xi \omega = \omega \longleftrightarrow \omega = d(\iota_\xi \omega)$,
ξ outwards at ∂U.



reverse time, or look at what points don't escape under ξ.

easy to see this limit set has Hausdorff dimension at most $1/2 \dim U$.

Different choices of v.f. correspond to diff. choices of spine, e.g.



vs.

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singularity is "good" if $\exists$ U Liouville s.t. $-\xi$ contracts to L.

ξ doesn't have to be trivial on L.

[Let's assume for now L has a nice finite Whitney strat.].

on L (w/ good stratification), $\exists$ natural cosheaf in homotopy sense of finite type dg-~~algebras~~ categories $/\mathbb{Z}$.

What does this mean & how to calculate?

Finite simplicial set S.

<u>Diagram of dg categories</u> $/S$ means:

$\forall\ s \in S$ $\quad C_s$ dg-cat.

$\forall$ non-deg. $s_0 \longrightarrow s_1$ $\quad$ dg-functor $F_{s_0 s_1}: C_{s_0} \to C_{s_1}$

$$\begin{tikzcd} & s_1 \arrow[dr] & \\ s_0 \arrow[ur] \arrow[rr] & & s_2 \end{tikzcd}$$

$\exists$ quasi-iso. of $F_{s_0 s_1} \cdot F_{s_1 s_2} \simeq F_{s_0 s_2}$

3-simplices $\rightsquigarrow$ homotopy between isomorphisms.

etc.

One can imagine a notion of homotopy limit here.

Claim: if all the dg-cats are of finite type, so is hol.im.

each C_{s_i} is:

finite quiver Q. arrows ordered, $\alpha_1, \alpha_2,$ have grading $\in \mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$

path category.

$d(\alpha_i) \in P(\alpha_{<i})$.

If so, this htpy limit is again of the same nature.

Diagram of points

S_0

1) dg cat. objects $= S_0$, freely generated by non-degenerate simplices of dim > 0.

$(i_0, \ldots, i_n)$ gives $\alpha_{i_0,\ldots i_n} : i_0 \to i_n$ of deg. $1-n$.

$$d\alpha_{i_0\cdots i_n} = \sum_{j=1}^{n-1} (-1)^{j-1} \left((\alpha_{i_0\cdots i_j} \cdot \alpha_{i_j \cdots i_n} + \alpha_{i_0 \cdots \hat{i_j} \cdots i_n} \right).$$

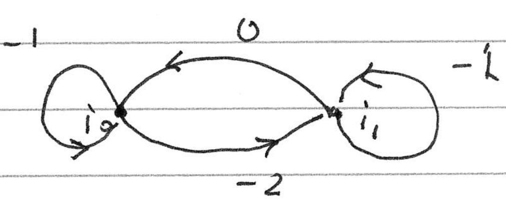
and

$\forall \ (i_0, i_1)$,

postulate that $\alpha_{i_0 i_1}$ is an isomorphism.

so want ~~cone~~ $\mathrm{Cone}(\alpha_{i_0, i_1}) \simeq 0$, can do this explicitly, get 2×2 matrix from cone, $(\mathrm{id}(\mathrm{cone}) = d(\mathrm{something})?)$.

need to add



Differential of each generator is very explicit

("Localization of the cone").

In the end,

We get a dg algebra $\mathrm{Chains}(\Omega(|S|, \text{base pt.}))$.
based loops

what do we mean by cosheaf of dg-cats?

For reasonable open sets, can go from small to large, ~~can calculate~~ with a cover, can calculate everything w/ appropriate hocolim over simplicial set.

— why should any singular lag'n mfld be such a gadget?

Call this cosheaf Φ_L

Firstly, what are ~~stalks & sections~~ stalks & global sections?

$\Phi_L(L)$ if L is compact should be also a $\mathbb{Z}/2$-graded dg-category of finite type with CY-structure.
(bit more than V. Ginzburg's CY structure).

$\mathcal{C}$ smooth dg cat
CY structure is an elt. in $HC^-(\mathcal{C})$ even or odd satisfying non-deg. in the following sense:

$$\begin{array}{cc} HC^-(\mathcal{C}) & \alpha \\ \downarrow & \downarrow \\ HH(\mathcal{C}) & \tilde{\alpha} \end{array}$$

$\tilde{\alpha}$ should be a quasi-iso. of bimodules.

If $\mathcal{C} = \mathrm{Perf}(A)$, this $= R\mathrm{Hom}(A^\vee, A)$,
where $A^\vee = R\mathrm{Hom}_{A\text{-mod-}A}(A, A\otimes A)$

When L is smooth, oriented, spin structure,

(*) $\Phi_L(L) \simeq \mathrm{Chains}(\Omega(L, x_0))$, in general different if not oriented or no spin str. chosen

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Rmk: $\forall$ diagram of $\mathbb{Z}/2$-dg cat $(S_\bullet$ changes to $\mathbb{Z}$ of cat. $\mathbb{Z}$-graded.
twist it,

$H^1(|S_\bullet|, \mathbb{Z}/2)$, b/c. $\mathbb{Z}/2 \to \mathrm{Aut}(\forall\, \mathbb{Z}/2\ \mathrm{dgcat})$ by shift functor

$H^2(|S_\bullet|, \mathbb{Z}/2)$ mult. of isomorphisms of functors.
$\sim \{\pm 1\} \in k^*$

for diff. reasons.

In (*), w/o spin/orientation, $\Phi_L(L) = \underrightarrow{\mathrm{holim}}$ (diagram of pts., twisted by $w_1(L)$, $w_2(L)$, Stiefel-Whitney classes)

easy to compute $w_1(L)$, $w_2(L)$ explicitly, given Barycentric subdivisions of simplices on L.

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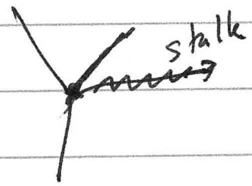
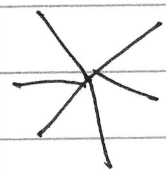
what to do at singularities?

triple pt.

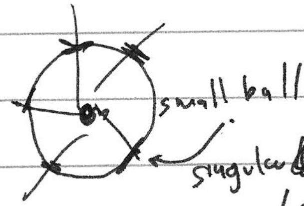


representations of quiver A_2



n-gon $\rightsquigarrow$ A_{n-1}

In general, guess:



stalk should be FS category of Lagrangian in ball w/ boundary on this Leg. don't really know how to do this.

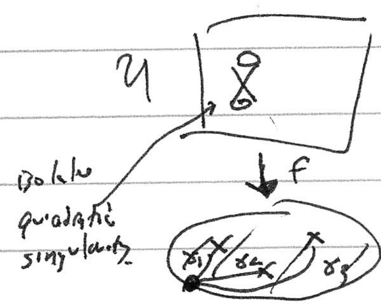
use some sort of induction calculation.

use Reeb flow for homs

$\Phi_L(L)$ for compact L.

$\mathcal{U} \supset L$

Liouville



locally trivial on $D - \{x_i\}_1^k$

$\mathcal{U}' = f^{-1}$(boundary point)

$\Phi(\mathcal{U}')$ finite type dg-alg. by induction, CY.

$\mathcal{E}_1, \ldots, \mathcal{E}_k$ spherical objects from $x_1, \ldots, x_k$, $\gamma_1, \ldots, \gamma_k$.

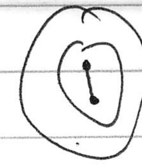
finite dim'l representations of $\Phi(\mathcal{U}')$?

FS category:

saturated w/ exceptional collection $\tilde{\mathcal{E}}_1, \ldots, \tilde{\mathcal{E}}_k$

$R\mathrm{Hom}(\tilde{\mathcal{E}}_i, \tilde{\mathcal{E}}_j) = 0$ $i<j$, id $i=j$.

n.b. boundary is allowed on L, e.g.



& $R\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j)$ same as $\phi(\mathcal{U}')$ if $i > j$.

Call this $\mathcal{FS}(\mathcal{U}, f)$.

because of $\mathcal{E}_1, \dots, \mathcal{E}_n$, get a natural ~~transformation~~ from

Serre functor

$$S_{\mathcal{FS}(\mathcal{U},f)} \to \mathrm{Id}.[\tfrac{\dim}{2}].$$

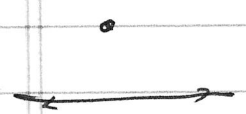
One should kill image of the cone of this map.

i.e. Localize w.r.t. ~~Cone~~ Cone (Serre $\to$ Id).

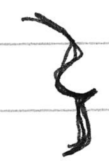
Main question: prove cosheaf property for this ~~story~~ local category

Rmk: stalks not generally CY, but global sections ought to be.

$\phi(L)$ D^b(coh Schemes)



$\mathbb{R}^n$.

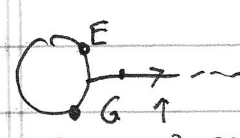


pt

pt.

why? alg. is chains on based loops, group rings of fund. gp., Laurent polys.

$\mathbb{A}^1 \setminus 0$



non-spcl. ray.

Claim: $\rightsquigarrow$ $\mathbb{A}^1$

i.e. get just polys, not Laurent polys.

$E \to F \to G$

exact triangle.

Say then that $E = F$.

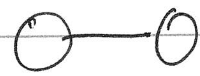
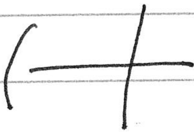
$\rightsquigarrow$ should correspond to $(\mathbb{C}^*, x + \frac{1}{x})$.

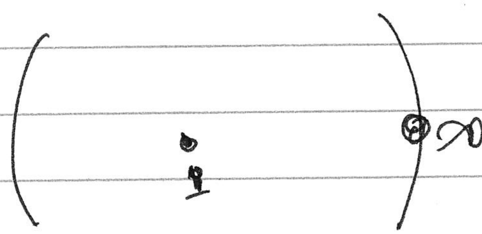
(*)  $\rightsquigarrow$ $\mathbb{P}^1$ or rep.  kronecker.

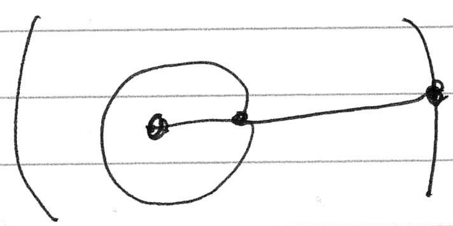
get exactly two morphisms between E, F.

[In general, 3-valent graphs $\rightsquigarrow$ get a category on which Teichmüller group acts

can shift pictures around]

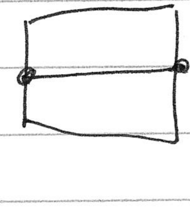
Interesting:  $\rightsquigarrow$ D^bCoh().

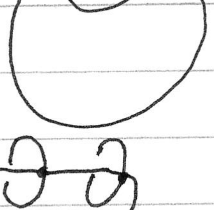
(*): $\mathbb{C}^*$:  contract $\mathbb{C}^*$ fixing $1, \infty$ gives us exactly:

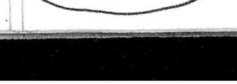
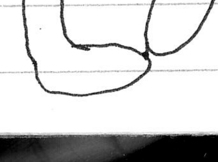


In general, potential will give us a non-cpt Lagrangian.

$\mathbb{C}, x^2$.



 $\rightsquigarrow$ $\mathbb{P}^1/_{0=\infty}$

 $\rightsquigarrow$ chain:  of $\mathbb{P}^1$'s

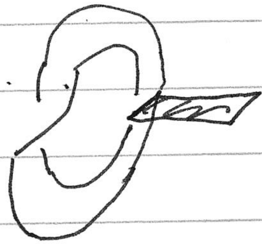
Get not all mflds, mflds defined over $\mathbb{F}_2$, combinatorial objects.

$\mathbb{P}^1 - \{0, 1, \infty\}$

$L \subset \mathbb{R}^6$, cone over $(S^1)^2$

unusual:

$\mathbb{Z}[x]/x^2$



Mobius band w/ glued strip?

~~what to~~ what to do w/ this? How to apply to real life?

X cpt. symplectic ~~mfld~~ ~~mflds~~ fold. $\omega = c_1(L)$ some L

(in some sense can make sections of $L^{\otimes k}$, $k \gg 0$)

Donaldson: $X \supset D$ (zeroes of section of L)

s.t. ~~$X - D$ is Liouville~~

$X - U_\varepsilon D$ is Liouville.

Contract to some spine L.

Get

$\Phi(L)$ finite type dg-cat. "wrapped Fukaya category for L", CY.

Seidel: try to reconstruct everything looking at this category

Ambient manifold

$\rightsquigarrow$ solutions of Maurer-Cartan in dg-algebra (namely)

$C^{cyc}(\Phi(L)) \hat{\otimes}\, q\, \mathbb{Q}[q]$

(boundaries of holo. discs in L).

deformations of $\Phi(L)$ which in certain sense should become saturated, non-cptness goes away.

Then we could get GW invariants!

to get GW invariants:

Claim: if C is smooth dg category, CY, then have a map

moduli of curves w/ tangent vecs. @ marked pts

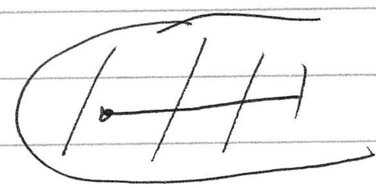
$$HH(C)^{\otimes n} \otimes H_\bullet(M_{g,\vec{n}+\vec{m}}) \to HH(C)^{\otimes m}$$

$n \geq 0,\ m \geq 1,\ g \geq 0.$

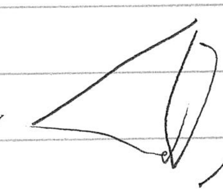
+ Action on the level of chains.
(similar to Yan's work on finite type stuff).

If degeneration of Hodge-de Rham, can put $\bar{M}$ and get whole story
(b/c this ~~...~~ implies S^1 action is trivial, so can glue & mod out by S^1's to get nodes)
(cpt Lagrs survive, give class in CC(fin. dim. reps), give Fuk of cpt mfld?).

entire preimage of



is a bundle around it



so fibrewise wrapping built in?