Kontsevich II — mixed nc motives

$HP_{/k}$ — right analogue of de Rham cohom. — base field, char $k = 0$.

? K_{top} $/\mathbb{Z}$? $K_{top} \otimes \mathbb{C} \simeq HP$

$k = \mathbb{C}$

comparison (want to use structure from varieties but problem: in some (non-alg. settings), not enough objects if one quantizes)

Connes' $\int$

suppose A assoc. $/ \mathbb{C}$

analogue of "topological cycle"

$A \to \mathrm{End}\, \mathcal{H}$ Hilbert

$F \in \mathrm{Aut}(\mathcal{H})$, $F^+ = F$, $F^2 = 1$

Fredholm module

$\forall N \in \mathbb{N}$ — Trace Re? Lit: Trea

$\forall a$: Trace $|[a, F]|^N < \infty$

periodic cyclic chain

$\sum a_0 \otimes \cdots \otimes a_{2n} \, u^n$ $\quad b + uB$

$\downarrow$

$$\frac{Tr(a_0 [F, a_1] \cdots [F, a_{2n}])}{(2\pi i)^n}$$ for $\forall n > N/2$.

Ex: $A = A_0 \otimes \mathbb{C}$. $A_0 / \mathbb{Q}$.

functional on $HP(A_0)$

$\mathbb{Q}$-vect space $\longrightarrow \mathbb{C}$.

one can show they're rigid, & can get all periods of alg. varieties over $\mathbb{Q}$.

Q: can we get new numbers or not? Not clear.

BIG ?: Develop a non-comm. tool for the comparison isomorphism.

(Rmk: this is definitely an infin.-dimensional phenomenon).

Example meant here: $A = C^\infty(X) \otimes \mathbb{C}$; ... periodic cyclic chain

module: spinor bundle S_{pi}

$\mathcal{H} = L^2(S_{pi})$

$F = \mathrm{sgn}(D)$ ← Dirac operator

$D \cdot \sqrt{D^2}^{\,-1}$.

Ex: $\displaystyle\int_{|x|=1} \frac{dx}{1 - \text{polynomial in } x}$

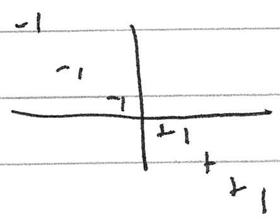
Calculate residue, get diff'l form $a_0\, da_1$:

$a_0 = \dfrac{1}{1-p(x)}$; $a_1 = x$,

Consider

$\mathcal{H} = L^2(S^1)$

$x^n \quad n \in \mathbb{Z}$



Calculate trace, get the residue.

point: trace is an infinite sum.

With this in mind, what happens for saturated dg-categories? pure Hodge structure.

firstly, $\dim HP^n < \infty$. e.g. if $\mathcal{C} = D^b(\mathrm{Coh}\, X)$, smooth, proper

(Conj): HP^n pure Hodge str. of weight n $\quad HP^\cdot = H^\cdot_{dR}(X \times \mathbb{P}^\infty)[u^{-1}]$

(going from n to $n+1$ or 2 some Mixed Tate business?) $\quad$ module $/ H^\cdot(\mathbb{P}^\infty)$ $= \mathbb{Q}[[u]]\beta$

Mixed nc-motives

- Ad hoc def'n: (∞) category enriched over spectra. (Mot^{pure})

Objects: saturated dg cat / given k (char $k = 0$) (k field)

Morphisms: $\mathrm{Hom}(C_1, C_2)$
$:= K^{alg}$-theory spectrum of $\mathrm{Fun}(C_1, C_2)$
$C_1^{op} \otimes C_2$.

what's K-theory spectrum? Same as connective K-theory spectrum.
can give a simple description as a sset.

SSet: n simplices are families of paramedized by $\mathbb{A}^n$.
(e.g. 1 simplices are $\mathbb{A}^1$-htopies).
& take geom. realization of SSet.

[K groups are up to torsion higher Chow groups, so agrees w/ usual motives in comm. case]

Can always view this category as a full subcat. of a triangulated cat,
by taking twisted complexes

mixed nc-motives := triangulated envelope + Karoubi closure
(direct summands?).

G. Tabuada: gave an explanation of this category: He considered
($2,\infty$) category of all small ~~dg cats~~ Δ-ted, Karoubi-closed, dg cats.
e.g. functors only up to isomorphism.

$\downarrow \infty$ functor $\mathcal{H}$. ← cohomology theory

($2,\infty$) triangulated category.

want:
(1) $\underline{0} \longrightarrow 0$

$\exists$ quotient cats

$$A \hookrightarrow B \longrightarrow B/A \quad \overset{\text{want: (2)}}{\rightsquigarrow} \quad \mathcal{H}(A) \longrightarrow \mathcal{H}(B) \longrightarrow \mathcal{H}(B/A)$$

full subcat: $A \to 0 \to B/A$ — composition factors through 0.

exact triangle

~~univ~~ universal cohom. theory $\mathcal{H}$ satisfying this.

(3) commute w/ filtered colimits

He proved: for $\mathcal{H}_{univ}$, then

$$\mathrm{Hom}(\mathcal{H}(\hat{k}), \mathcal{H}(c)) = \text{non-comm. } k\text{-theory spectrum}$$

- k-theory pops out of Karoubi closed & notion of localization
- Kontsevich's mixed nc-motives is some sort full subcat.

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examples: Hochschild homology., & global fns. on an alg. var. for same reason (unnatural, needs to be excluded somehow).

Mixed nc-motives $\xleftrightarrow[\text{full embedding}]{}$ Voevodsky mixed motives

$\otimes\mathbb{Q}$

$/ \otimes \cancel{H^2(\mathbb{P}^1)[-2]}$.

$H^\cdot(\mathbb{P}^1, \mathbb{P}^0)$.

(b/c 2-periodic up to twist by Tate motive?)

M_1, M_2 usual motives

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Ext}^\cdot(M_1, M_2 \otimes \quad \otimes n).$$

How to construct objects ~~___~~ in mixed nc motives?

1° twisted complexes

objects are categories, add shifts, define direct sums of shifts (upper triangular way),

$$\partial \quad 0 \sim C^i \xrightarrow{F^i} C^{i+1} \xrightarrow{F^{i+1}} C^{i+2} \to \text{--} \quad 0 \text{--}$$

h_0, k_1

no arrows going back b/c k-theory non-neg!

saturated categories

: deformation of $\bigoplus C^i[i]$

$F^i: C^i \to C^{i+1}$ functor (point in k-theory?)

$F^{i+1} \circ F^i \sim 0.$ — for any triple, get $\mathbb{A}^2$ homotopy, etc. — finite amt. of data.

$\mathbb{A}^1$ htpy ↑ could be a chain of $\mathbb{A}^1$ htpies.

$\partial^2 = 0$ on the nose also ok.

Another way: finite diagram of dg categories, functors strictly associate.

X smooth proper ... $HP^n = H^n(X \times \mathbb{P}^\infty)$ $n \gg 1$.

Simplicial set S.

$\alpha \in S_0$ C_α, translate this one to another ...

$\bigoplus CC^-(C)[i]$ negative cyclic complex(es). complex of saturated categories

- total differential.

get $HC^-(--)$

Conj: is free $k[[u]]$-module over $H_{dR}(\mathbb{P}^\infty)$.

degen. of Hodge to de Rham.

(e.g. variety w/ divisors w/ normal crossing, cohomology of pairs).

Simple examples: X smooth proper $\supset D_1, D_2$ smooth divisors

$D_1 \cap D_2 = \emptyset$.

$$0 \to D^b(\mathrm{coh}\, D_1) \xrightarrow{(i_1)_*} D^b(\mathrm{coh}\, X) \xrightarrow{(i_2)^*} D^b(\mathrm{coh}\, D_2) \to 0$$

deg: -1 $\quad 0 \quad 1$

comp. $= 0$ on nose b/c no intersecting.

HP is something like cohom. of pair: $H^\bullet(X - D_1, D_2)$.

e.g. $X = \mathbb{P}^1$, $D_1 = 2$ pts. $D_2 = 2$ pts then get a mixed H.S. given by logarithm

Q: is there a way of resolving a non-saturated cat. by saturateds?
e.g. open variety. Don't know.

From this mixed story one can deduce several things:

object $\in C_{\text{saturated}}$ is a functor:

$0 \to \bullet \longrightarrow C \to \supset$, already get a complex, take invariants.

then get maps $K_0^{\deg}(C) \to \pi_0(\mathrm{holim}(K_{top}(C) \to HP(C) \leftarrow HC^-(C)))$.

get something like:

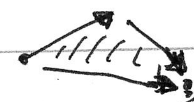
$$0 \to HP^1(C) / (F^1 HP^1 + K^1_{top}) \to H^0_{Deligne}(C) \to \underbrace{K_{top}^0 \cap F^0 HP}_{\text{"Hodge classes"}} \to 0$$

torus of intermediate Jacobian

"Hodge classes" (p,p) type classes

normal mflds:

$H^{odd}_{dR}(X) / \bigoplus_{p \geq q} H^{pq} + H^1_B(X, \mathbb{Z})$ (Look at back).



A-model:

X_i $\qquad$ X_{i+1}

X_i = symplectic manifold $= \bigcup X_{i,\alpha}$.

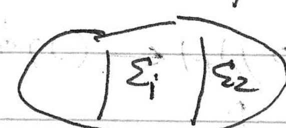
symp. mfld of same dimension, cpct.

for any setup w/ fixed maps, get an elliptic problem

twisted complexes, etc.

$\rightsquigarrow$ holom. curves

Σ surface $\quad \Sigma = \coprod \Sigma_i$



$\Sigma_i \to X_i$ holom.

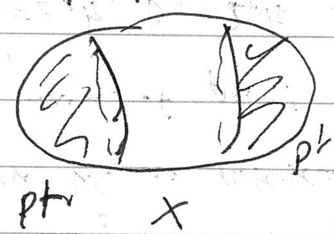
boundary $\to$ Lag'n. (e.g. Woodward–Wehrheim)

Exercise: how to interpret periods of mixed H.S. in Fuk.-cat. setup.

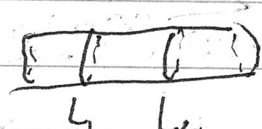
e.g. $L_1, L_2 \subset X$, $L_1 \cap L_2 = \emptyset$.

$\rightsquigarrow$ pt. $\to X \to$ pt., composition $= 0$.

count



X non cpct



count cylinder w/ boundary L_1, L_2. There is one cylinder but also n-fold coverings, which have ~~n-fold~~ order n automorphism.

$\rightsquigarrow$ get $\sum \frac{1}{n} e^{-(\text{area of cyl.}) \cdot n}$ ← logarithm, really a period of mixed H.S. associated to this situation.