Miami Mirror Symmetry '10

Abouzaid I: A Functorial Point of View on HMS

X smooth complex variety
D anticanonical divisor, $\sigma \in K_X^{-1}$, $\sigma^{-1}(0) = D$.
$X \setminus D$ open CY with volume form $\Omega = \sigma^{-1}$.
also pick a symplectic form on X, compat. w/ complex structure?

Strominger–Yau–Zaslow (FOOO, Auroux, —)

Conj: $\exists$ SLag torus fibration

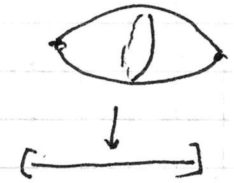
$$\begin{array}{ccccc} T^n \Rightarrow F_b & \to & X & \supset & D \\ \downarrow & & \downarrow & & \downarrow \\ b \in & & \overline{B} & \supset & \partial \overline{B} \end{array}$$

where b lies away from a codim 2 subset of $\overline{B}$.

$\overline{B}$ is a manifold w/ boundary (corners?) w/ $\underbrace{\dim_{\mathbb{R}} = \dim_{\mathbb{C}} X.}$

over $\partial\overline{B}$, we have T^{n-1} torus fibration. (original SYZ).

dim 1: $X = \mathbb{P}^1$, $D = \{0, \infty\}$, There's a projection: $z \longrightarrow x$, $\log|z| \in [-\infty, +\infty)$

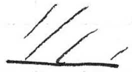


dim 2: Locally $\mathbb{C}^2 \to [0, +\infty)^2$, $(z, w) \mapsto (|z|^2, |w|^2)$

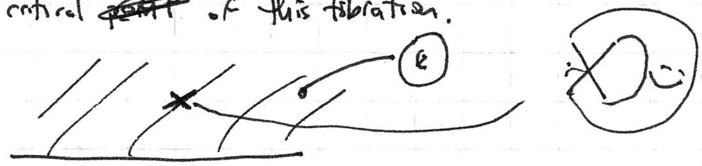
$D = \{zw = 0\}$

Deform this to a fibration $\mathbb{C}^2 \to \mathbb{R} \times [0, +\infty)$, $(z, w) \mapsto (|z|^2 - |w|^2, |zw - \varepsilon|^2)$.

$D = \{zw = \varepsilon\}$



At $(0, \varepsilon^2)$ we have a critical ~~point~~ value of this fibration. (crit. point is $(0,0)$)



If you ignore the "special" part of the notion of the fibration, then you can patch these things together.

E.g. $\mathbb{CP}^2 \xrightarrow{\text{moment map}}$; $\mathbb{CP}^2$ deform $\rightsquigarrow$ $\mathbb{CP}^2 \to$

← not apparent but bdy. should be totally geodesic w.r.t. affine str.

(Q: do we also have Monge–Ampère eqns. solutions?)
A: Yes. Some nice ~~metric~~? to reconstruct sympl/complex guys. Depends on affine str. Analogue exists, hasn't been studied at all).

Strategy due to Gross & Siebert for producing these fibrations from toric degenerations –

E.g., take complete intersection, degen. to hyperplanes. (need semiflatness to get a metric on the base you can lift!)

<u>SYZ conj</u>: the mirror of (X, D) is obtained by "dualizing" this fibration.

$B = \overline{B} - \partial\overline{B}$.

E.g. $Y \sim F_b^\vee := \begin{cases} \bullet\ U(1) \text{ local systems on } F_b \\ \bullet\ H^1(F_b, \mathbb{R}/\mathbb{Z}) \end{cases}$.

$\downarrow \qquad \downarrow$

$B \ni b \leftarrow$ reg. value of fibration

together with the superpotential $W : Y \longrightarrow \mathbb{C}$ which counts holomorphic curves with ∂ on the fibers.

$$W(b, \nabla) = \sum_u e^{-\int u^*\omega + i\,hol_\nabla(\partial u)}.$$

The problem of constructing ω_Y (sympl. form) from the complex structure on X is local.
But, constructing the complex structure on Y is NOT local $\rightsquigarrow$ Kontsevich & Soibelman wall-crossing, (b/c depends on Floer theory on other side, non-local). 2001.

B-side on X | A side

$\mathrm{Coh}(X) \rightsquigarrow \mathcal{F}uk(Y, W) \leftarrow$ Fukaya cat. of superpotential.

Objects are Lag. in Y whose image under W =

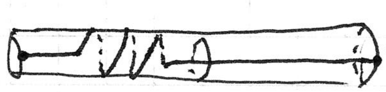
$\mathrm{Coh}(D) \rightsquigarrow \mathcal{F}(W^{-1}(p))$, $p \gg 0$.

(true if D smooth, if not, need to make sense of this $\nearrow$ careful).

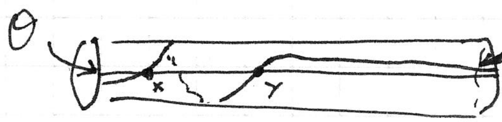
$\mathrm{Coh}(X \setminus D) \rightsquigarrow \mathcal{W}(Y) \leftarrow$ objects are non-compact Lagrangians, technically maybe well-behaved at ∞.

Instead of defining these, let's see what happens in the simplest case

<u>ex</u>: $\mathbb{CP}^1 \rightsquigarrow W = z + \frac{1}{z}$ on $\mathbb{C}^*$



$\boxed{\mathrm{Coh}(X)}$ $\mathcal{O}(i) \longleftrightarrow$ {Isotopy classes of lines from $-\infty$ to $+\infty$} $\sim \mathbb{Z}$ (winding #).

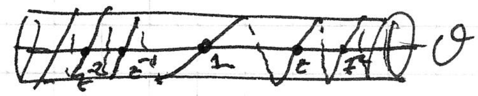


$\mathrm{hom}(\mathcal{O}, \mathcal{O}(1))$. $\rightsquigarrow$

$\boxed{\mathrm{Coh}(D)}$ trivial here, so skip.

$\boxed{\mathrm{Coh}(X \setminus D)}$

$\mathbb{C}^* \longleftrightarrow$ Objects are circles & lines, but all lines are isotopic (can move endpoint).



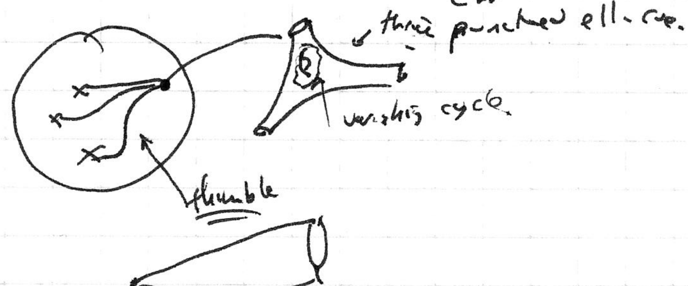
$\mathbb{C}[z, z^{-1}] \xrightarrow{\sim} HW^*(\theta, \theta)$

Can also check multiplication works out, w/ triangles

Now, $\mathbb{CP}^2$ with $D = \{xyz = 0\}$.

Mirror is $(\mathbb{C}^*)^2$ with $W = z + w + \frac{1}{zw}$

three critical points



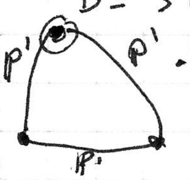
The category generated by thimbles is called $\mathcal{FS}(W) \hookrightarrow \mathcal{F}(W)$ defined before (has anyone proved this embedding is everything?)

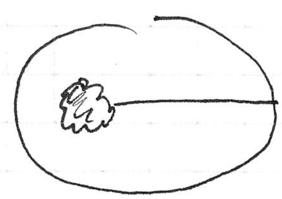
Seidel checked that this is mirror to an exceptional collection on $\mathbb{CP}^2$.

- Mirror to D is 3-punctured genus 1 surface $= \Sigma$

$\mathrm{Perf}(D) \rightsquigarrow \mathcal{F}(\Sigma) \leftarrow$ cpt. Lag'ns.

$D \simeq 3\,\mathbb{P}^1$'s.





<u>Claim</u>: $\mathrm{Coh}(D) \rightsquigarrow \mathcal{W}(\Sigma)$

$X \setminus D \simeq (\mathbb{C}^*)^2 \overset{\text{mirror}}{\longleftrightarrow} (\mathbb{C}^*)^2 = T^*T^2$

$\mathbb{C}[z^{\pm 1}, w^{\pm 1}] \longleftrightarrow HW^*$ (pos. reals in $\mathbb{C}^{*2}$) $(\mathbb{R}^+)^2$.

At the level of objects, clearly a map

Claim: $\exists$ A_∞ functors $\mathcal{F}(W) \xrightarrow{(*)} \mathcal{W}(Y)$ ← acceleration

[Joint in progress w/ Seidel]

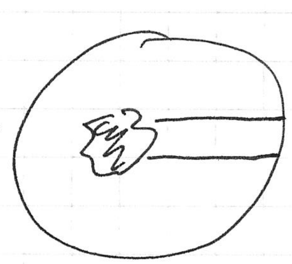
$r \downarrow (**)$ ← clear on objects for $p \gg 0$.

$\mathcal{F}(W^{-1}(p))$ ← restriction.

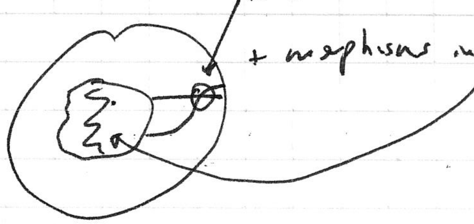
Idea: construct a model for $\mathcal{F}(W)$ s.t. $\mathcal{W}(Y)$ sits as a quotient category.

Words about (**):

two objects



morphisms $\rightsquigarrow$



intersections in fibre $W^{-1}(p)$,
+ morphisms in } $\cong$ a quotient subcomplex if you arrange it nicely

How to use these A_∞ functors

Assume that W is a Lefschetz fibration. (i.e. all singularities are of Morse type + ε). nice conditions @ ∞.

$FS(W) \hookrightarrow F(W)$. It's a directed A_∞ category.

Have this functor
$$\begin{tikzcd} FS(W) \arrow[r, "r"] \arrow[dr, "r"'] & F(W^{-1}(0)) \arrow[d, phantom, "\cup"] \\ & \mathrm{Image}(FS(W)) =: \mathcal{B}. \end{tikzcd}$$

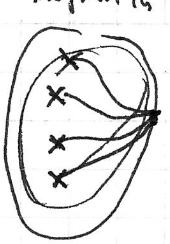
Have an adjoint
$$r^* : \mathcal{B} \longrightarrow Perf(FS(W))$$
(in general, all modules, but directed gives us this)

(really, $D^\pi(\mathcal{W}(Y)) \simeq D^\pi(FS(W)/r^*(\mathcal{B}))$.

Claim: $Perf(\mathcal{W}(Y)) \simeq Perf\left(FS(W)/r^*(\mathcal{B})\right)$

Strategy: (0) Prove that thimbles split generate $\mathcal{W}(Y)$. (tomorrow)
(1) Compare morphisms in $\mathcal{W}(Y)$ & quotient for thimbles.

Morphism in $\mathcal{W}$



$\rightsquigarrow$ Bending a thimble around many times. So analyze what happens when you bend once.

Geometrically, the Serre functor of $FS(W)$ corresponds to "bending" once (parts of the proof by Seidel).

There exists a natural transformation from Serre $\to$ Id (shift) coming from counting sections

This is mirror to:

$D^b Coh(X \setminus D)$ is loc. of $D^b Coh(X)$ at $D^b Coh_D(X)$ supported along D.

Key: $Fs = \mathcal{A}$, $\mathcal{B} = \underline{\text{full}}$ image under r.

$$Fs \xrightarrow{r} \mathcal{B}.$$

$$r^*: \mathcal{B} \to \mathcal{A}\text{-mod}.$$
$$r^*\mathcal{B} := \mathcal{A} \otimes \mathcal{B}/\mathcal{A}.$$

<u>Barannikov – Developments in the ~~BV~~ non-comm. BV formalism</u>

$S_{0,1}$ – A_∞-alg. w/ (even/odd) scalar product, so S – multiloop higher genus generalization of A_∞-alg.

$$\hbar \Delta S + \tfrac{1}{2}\{S, S\} = 0$$
$$\Leftrightarrow \Delta(\exp \tfrac{1}{\hbar} S) = 0$$

solutions to nc BV-eqn have characteristic homology classes in $H_*(\overline{\mathcal{M}}_{g,n})$,

not Deligne–Mumford compactification, Kontsevich compactification.

answers question of M. K.: find alg. str. extending Massey products of A_∞-alg. in $H_*(\mathcal{M}_{g,n})$ to $H_*(\overline{\mathcal{M}}_{g,n})$. + Witten – combin. model for compactification of $\mathcal{M}_{g,n}$.

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higher genus analogue of theory of variations of (nc) Hodge str. (of CY-type)

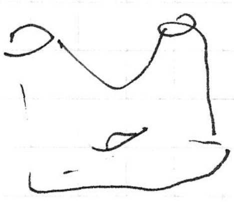
A_∞-periods, nc-VHS $(HC_*^- \subset HP) \to (H_*(\overline{\mathcal{M}}_{0,n})\text{-action})$ on HH

- A_∞: $\sum$ trees $\longleftrightarrow$ nc-BV: $\sum$ graph

Feynman transform of "modular operad" – Kapranov–Getzler.

~~[illegible]~~ $\dim M/2$ even: $\Lambda(\mathbb{C}^n)$ $\left[\text{sol'ns to: } \Delta(\exp \tfrac{S}{\hbar}) = 0\right]$ ← BV eqn.

$\dim M/2$ odd: $S(\mathbb{C}^n)$



$S_{g,i}$ = # (genus g holom. curves w/ i bd. components)

$(\Sigma, \partial\Sigma, p_i) \to (M, L_i, \oplus L_i \cap L_j)$

vector space is $\oplus H_*(L_i \cap L_j)$

Not proved