Kontsevich 3 Hol. Chern Simons + ...

Today: 3-dim. CY categories.

$$\mathcal{E} \rightsquigarrow \quad m_n : \underline{\mathrm{Hom}}(\mathcal{E},\mathcal{E})^{\otimes n} \to \underline{\mathrm{Hom}}(\mathcal{E},\mathcal{E}) \quad \deg. 2-n$$

$(n \geq 1)$

d dim. CY : $(,) : \underline{\mathrm{Hom}}(\mathcal{E},\mathcal{E}) \otimes \underline{\mathrm{Hom}}(\mathcal{E},\mathcal{E}) \to k$ deg $-d$.

combining: $(m_n(- -),) : \mathrm{Hom}(\mathcal{E},\mathcal{E})^{\otimes(n+1)} \to k \qquad 2-d-n$
$= (3-d)-(n+1)$

cyclically symmetric (w/ some sign?)

Deform: m_0 can appear.
when $d=3$, ~~...~~ can have $m_{-1} \in k$ for $d=3$ (same sign.)

what is this number? hol. Chern Simons

Objects of "right" 3CY
= critical points of a function
m_{-1} = critical value.

X alg. variety $/\mathbb{C}$
$f : X \to \mathbb{C} \qquad f \in \mathcal{O}(X)$

finite subset $\mathrm{Bif}_f \subset \mathbb{C}$ Bifurcation values (NOT crit. val.)
$= \{z_1, \dots, z_n\}$

$z_0 \notin \mathrm{Bif}_f$ if $f|_{f^{-1}(\text{nbhd})}$ trivial fibration (If f proper, same as crit. values, but in general larger)

$\rightsquigarrow$ constructible sheaf of $\mathbb{Z}$-mod $/\mathbb{C}$ (abelian gps.)
$n \in \mathbb{Z}$, s.t. $R\Gamma(\mathcal{E}) = 0$

stalk $\mathcal{E}_z = H^n_B(X, f^{-1}(x); \mathbb{Z})$

Betti cohom.
(can use any kind of cohom. theory)
$R\Gamma$ will be 0, specific to $\mathbb{Z}$ dim.

~~Pastro-B~~ uk(P?)

$\longleftrightarrow V_1, \ldots, V_n$ $\mathbb{Z}$-mod

$\forall i, j$ get $T_{ij} : V_i \to V_j$ T_{ij} invertible

$$V_i = H^n(f^{-1}(\quad), f^{-1}(\quad); \mathbb{Z})$$

T_{ii} like monodromy

T_{ij}: onned:

z_i —— z_j

straight path

"wall crossing phenomenon"?

(as long as no 3 on same line)

fix V_i, T_{ii}.

~~Gen~~ choices of T_{ij}, $i \neq j$ Braid group acts

Consider $\mathfrak{g} = \bigoplus_{i,j} \mathrm{Hom}(V_i, V_j)$,

(root lattice of sl_n!

graded by $(0\ 0\ \underset{i}{-1}, \ldots \underset{j}{+1}, \ldots 0) \in \mathbb{Z}^{n-1}$

stability: $z_i \to z_j$

$\mathcal{E}$ is in fact a <u>Mixed Hodge Module</u> (M. Saito ...)

R-constructible, but in fact perverse -- i.e. concentrated in one degree 2?)

<u>filtered holon. $\mathcal{D}$-module</u> deRham cohom $= 0$

(consider sheaves with $R\Gamma = 0$

(Tannakian) symm. monoidal $\otimes$ category w/ $\otimes :=$ additive convolution

(can use $*$ or $!$, they coincide here)

Weight filtration (coming from mixed hodge modules ...).

This category has fibre functor :-

exponential motives $=$ MHM (mixed hodge modules) with $R\Gamma = 0$ important to not use cplx analytic topology.

fiber functors $H_B^n(X, f) = H_B^n(X(\mathbb{C}), f^{-1}(-\infty); \mathbb{Z})$

DeRham realization $H_{dR}(X, f) = H_{dR}(X_{Zar}, (\Omega_X, d + df))$ Hypercohom.

Notice $B\mathrm{ot}_R$ ~~#~~ not exactly = crit. pts.

Can define:

<u>Critical</u> cohomology:

sheaf of vanishing cycles.

$$\bigoplus_{\substack{z_i \text{ crit} \\ \text{values}}} R\Gamma(f^{-1}(z_i), \varphi_{f-z_i}(\underline{\mathbb{Z}}_X))$$

in general different, but in many cases the same.) but no record in literature of equality

De Rham realization is $/ \mathbb{C}((u))$

$$H_{dR}(X_{Zar}, (\Omega^\bullet_X((u)), d + \tfrac{df}{u})).$$

Also a notion of nc-Hodge str, related to first thing.

Main example: <u>Matrix integrals</u>

cyclic polynomial, NOT formal power series

Q finite quiver, $W \in \mathbb{C}Q / [\mathbb{C}Q, \mathbb{C}Q]$

vertices $\underline{1 \dots n}$

$$\gamma = (\gamma^i) \quad \gamma^i \geq 0$$

$$M_\gamma = \mathbb{C} \text{ rep. of } Q \text{ in } (\mathbb{C}^{\gamma_i})$$

Gauge group $G_\gamma = \prod GL(\gamma_i, \mathbb{C})$

$$W_\gamma = \mathrm{Tr}\, W \text{ in repr.} \in \mathcal{O}(M_\gamma)^{G_\gamma}.$$

Want $\int_{\substack{\text{rapid} \\ \text{decay} \\ \text{cycles}}} e^{W_\gamma}$ equivariantly, i.e. take

~~BG_γ = prod.~~

$$\mathbb{E}G_\gamma \times_{G_\gamma} M_\gamma$$

$$\downarrow$$

BG_γ – prod. of ∞-dim'l Grassmannians, approx. by fin. dim'l ones

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In physics, (do something different:) study matrix integrals, see dependence on dimension, take $\to$ w/r.t. dimension

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$$\mathcal{H}_\gamma = H_{G_\gamma}(M_\gamma, W_\gamma)$$

equiv. cohom. -

For counting: use weight filtration $\rightsquigarrow$ counting (ref. Soibelman

Stability: $Z_i \in \mathbb{C}$, $\operatorname{Im} Z_i > 0$, then can speak of

$$M_\gamma \supset M_\gamma^{(ss,\, Z_i)}$$ ← semistable reps. open part

Use ..., Harder-Narasimhan filtration to calculate whole algebra using semistable, etc.?

⇝ 3CY with t-structure

objects in heart is $\bigcup_\gamma \operatorname{Crit} W_\gamma$ (Reps. of Jacobian algebra).

Associate crit. cohom. H_{crit}, may be what we want?
(no Thom isom. here), after total integration, stuff cancels & still
get wall-crossing.

On objects of 3CY,
get sheaf of vanishing cycles
$R\Gamma$(stack of objects, coeffs in this sheaf) critical cohomology.

3CY categories: 2 classes

① A finite type dg algebra + 3CY structure (non-cpct. CY, objects
$\mathcal{C}$ = finite dimensional dg-modules w/ cpct. support)

② A proper dg alg., 3CY
$\mathcal{C}$ = perf A-mod.

Example: X 3 dim CY/$\mathbb{C}$ (non-cpct
$\cup$
Z closed subset in $\overset{\text{Zar. top.}}{\text{cpct}}$ $\mathcal{C} = \operatorname{Perf}_Z(X)$
(not subscheme) / what's A? Take a strong generator, take End dgs.

Q, W polynomial

W formal power series.

In general, hereditary alg. ~~of finite type~~ & fin. generators, formally smooth (kernel of mult. is projective?)

A $W \in A/[A, A]$. (path alg. of quiver??)
fin.-dim'l rep'ns, & crit. values.

map

$$K_0(\mathcal{C}) \longrightarrow \mathbb{Z}^n \dashrightarrow \mathbb{C}$$

stability
Z ← central charge

Pick arbitrary perfect

$P_1, \ldots, P_n \in \mathrm{Perf}(A\text{-mod})$

$\mathcal{E}$ fin. dim. $\to (\chi\, R\mathrm{Hom}(P_i, \mathcal{E}))$.

C^{ss} stack of finite type for any γ.

We need a constructible sheaf of vanishing cycles; does not come for free.

moduli stack of all objects.

normalized dimension mod 2. 2 choices in $H^0_{et}(\mathrm{Mod}, \mathbb{Z}/2)$ ← like w_1, w_2 for Fuk

det bundle / mod $\wedge 2$. $H^1_{et}(\mathrm{Mod}, \mathbb{Z}/2)$

critical cohomology, ← much nicer global object

Suppose now X smooth proper (CY 3-fold) / $\mathbb{C}$.

$\mathcal{E}$ ~~holom~~ trivial C^∞ vec. bdle.

$\downarrow$ connection

X $\rightsquigarrow A \in \Gamma(X, \Omega^{0,1} \otimes \mathrm{End}(\mathcal{E}))$

$$CS(A) = \int_X \mathrm{Tr}\left(\frac{A\bar\partial A}{2} + \frac{A^3}{3}\right) \wedge \Omega^{3,0}$$ ← some given 3,0 form.

get holom. fcn. on ∞, Gauge gp. $:= \mathrm{Aut}(\mathcal{E})$ ↺

Gauge gp. doesn't preserve CS, one id. component does, on other components adds const, so (d of this functional) is preserved.

$\pi_0(\mathrm{Aut}\,\mathcal{E}) \to H_3(X; \mathbb{Z})$

value of CS shifted by $\int \Omega^{3,0}$
class in $H_3(X, \mathbb{Z})$

More generally, $\mathcal{E}$ $\mathbb{Z}$-graded C^∞ bundle..

space of all {super connections} locally connections + $\Gamma(X, \Omega^{0,\bullet} \otimes \mathrm{End}(\mathcal{E}))^1$

all forms; components of deg. 1

On / get hol. closed 1-form.

If we Try to see sheaf of v.c. on this problem: inf-dim'l manifold.

If Remove extra variables, reduce to fin. dimensions. Requires answering an orientation question.

$\{\text{super conn.}\}$ / connected component of id of $\mathrm{Aut}(\mathcal{E})$ as $\mathbb{Z}$-graded bundle $\xrightarrow{CS} \mathbb{C}$
defined up to const.

$\downarrow$ $H_3(X, \mathbb{Z})$ cover

all connections

A stability condition in the sense of Bridgeland
↳ stack of finite type.

critical values

crystal type things,

gauge

real things

$$\left(\int \Omega^{3,0}\right)\left(H_3(X, \mathbb{Z}) + \text{finite } \& \text{ } t\right)$$

At each critical value, expect critical cohom.

And for any straight line, additional operator on H_3.

transcendental things relating two crit values

e.g. count gradient lines in Kähler moduli connecting one crit pt. to another, for potential

(Ref. Witten, Chern Simons for 3folds).

cross fingers & hope no singularities @ ∞.

so really solve some eqn. on 7-dim'l mfold (G2 mfolds).

gradient flows in space of objects is 7-dim'l. (e.g. $CY \times \mathbb{R}$)

Not part of the category theory → need to add G2 str.

Get wall crossings of two types, this & change stability condition, moduli appears & disappear

So get some complicated new cohomology coming from CS theory.

If $X = CY^3$, $D^b(\mathrm{Coh}(X))$, get classical theory of usual forms before this G2 addition, which is non-algebro-geometric.

• elliptic problem, analogue of Yang-Mills, self-dual...

These categories are like octonions. Instead of cohomology of v.c., need FS cat of ∞-dim'l objects.

~[FS cat has the wrong hodge str. 2 2].

Q: Does this correspond to quantum ~~??~~ ?

can have analogue of CS for A-model, dilogarithm etc.

Don't really know how to translate --

maybe a count of associative flows between crit pts on LCS's,