

Kontsevich 3 Hol. Chern Simons + ...

Today: 3-dim. CY categories.

$$E \rightsquigarrow m_n: H_n(E, E)^{\otimes n} \rightarrow H_n(E, E) \text{ deg. } 2-n$$

$$\text{dim. dCY: } (,) : H_n(E, C) \xrightarrow{m_n} H_n(E, E) \rightarrow k \text{ deg. } -d.$$

$$\text{combining: } (m_n(---),) : H_n(E, E)^{\otimes(n+1)} \rightarrow k \quad 2-d-n = (3-d)-(n+1)$$

cyclically symmetric (w/ some signs).

Defn: m_0 can appear. when $d=3$, ~~...~~ can have $m_{-1} \in k$ for $d=3$ (same as n .)

what is this number? hol. Chern Simons

Objects of "right" 3CY = critical points of a function
 m_{cp} = critical value.

$$X \text{ alg. variety / } \mathbb{C} \\ f: X \rightarrow \mathbb{C} \quad f \in \mathcal{O}(X)$$

finite subset $\text{Bif}_f \subset \mathbb{C}$ Bifurcation values (NOT crit. val.)
 $\{z_1, \dots, z_n\}$

$z_0 \notin \text{Bif}_f$ if $f|_{f^{-1}(z_0)}$ trivial fibration (if f proper, same as crit. values, but is general (larger))

\rightsquigarrow constructible sheaf of \mathbb{Z} -mod / \mathbb{C} (abelian grp.)

$$n \in \mathbb{Z}, \text{ s.t. } R\Gamma(E) = 0.$$

$$\text{stalk } E_z = H_B^n(X, f^{-1}(z); \mathbb{Z})$$

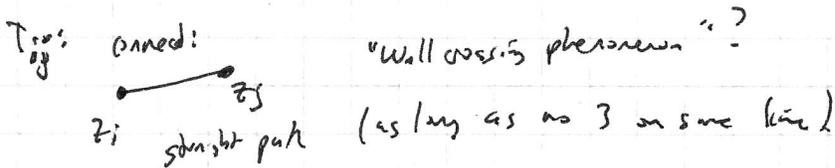
Betti class.
 (can use any index of char. theory)
 $R\Gamma$ will be 0, specific to 2 dim.

~~(WLP)~~
 $\longleftrightarrow V_1, \dots, V_n \mathbb{Z}\text{-mod}$

$\forall i, j$ get $T_{ij} : V_i \rightarrow V_j$ T_{ij} : invertible

$$V_i = H^n(f^{-1}(\text{circle with } z_i), f^{-1}(\text{circle with } \bullet)); \mathbb{Z}$$

T_{ij} : like monodromy



fix V_i, T_{ij} .

~~choices~~ choices of $T_{ij}, i \neq j$ Braid group set

consider $\mathcal{O}_Y = \bigoplus_{i,j} \text{Hom}(V_i, V_j)$

(root lattice of S_n !)

graded by $(0, 0, -1, \dots, +1, \dots, 0) \in \mathbb{Z}^{n-1}$

stability: $z_i \rightarrow z_j$

\mathcal{E} is in fact a Mixed Hodge Module (M. Saito - -)

proconstructible, but in fact perverse - - (ie concentrated in one degree 2?)

filtered holom. \mathcal{D} -module deRham cohom = 0

(works sheaves) with $R\Gamma = 0$

(Tannakian) \otimes category = / $\mathcal{Q} :=$ additive convolution (can use $*$ or $!$, they cancel here)

Weight filtration (coming from mixed hodge modules - -)

This category has fibre functor: = mixed hodge modules

exponential motives = MHM with $R\Gamma = 0$ important to not use cplx analytic types

fibre functors $H_B^n(X, f) = H_B^n(X(\mathbb{C}), f^{-1}(-\infty); \mathbb{Z})$

DeRham realization $H_{dR}(X, f) = H_{dR}(X_{\text{zar}}, (\mathbb{R}_X, d+dR))$ Hyperbolic

Note: B_{DR} ~~is~~ not exactly = cont. etc.
 Can do this:

Critical cohomology:

Proof of vanishing cycles.

$$\bigoplus_{z_i \text{ crit values}} R^1(f^{-1}(z_i), \mathcal{Y}_{f^{-1}z_i}(\underline{\mathbb{Z}}_n))$$

in general different, but in many cases the same.

De Rham realization is $\mathbb{C}((\hbar))$

but no record in literature of equality

$$H_{\text{DR}}(X_{\text{zar}}, (\Omega_X((\hbar)), d + \frac{d\hbar}{\hbar}))$$

Also a notion of $n\epsilon$ -body star, related to first thing.

Main example: Matrix integrals

cyclic polynomials, NOT formal power series

vector
 $1 \dots n$

\mathbb{Q} finite quiver, $W \in \mathbb{C}^{\mathbb{Q}} / [\mathbb{C}^{\mathbb{Q}}, \mathbb{C}^{\mathbb{Q}}]$

$$x = (x^i) \quad x^i \geq 0$$

$$M_x = \mathbb{C} \text{ rep. of } \mathbb{Q} \text{ is } (\mathbb{C}^{x^i})$$

Gauge group $G_x = \prod GL(x^i, \mathbb{C})$

$$W_x = \text{Tr } W \text{ in repr. } \in \mathcal{O}(M_x)^{G_x}$$

Want $\int e^{W_x}$
 rapid decay
 cycles

equivalently, i.e. take

~~$$B_{G_x} = \text{prod.}$$~~

$$\in G_x \times_{G_x} M_x$$

B_{G_x} = prod. of ∞ -dim'l Grassmannians, approx. by fin. dim'l ones

↳ something different:

In physics, study matrix integrals, see variance in dimension, take to $n \rightarrow \infty$ dimension

$$\mathcal{Z}_x = H_{G_x}(M_x, W_x)$$

← equiv. cohom.

For counting: use weight filtration \rightarrow counting local Betti numbers

Stability: $Z_i \in \mathbb{C}$, $\text{Im } Z_i > 0$, then can speak of

$$M_Y \supset M_X^{(SS, Z_i)} \leftarrow \begin{matrix} \text{upper part} \\ \text{semistable repr.} \end{matrix}$$

Use \dots , (Hodge-Narasimhan filtration \hookrightarrow cyclic whole algebra uses semistable, etc?)

\leadsto 3 CY with t-structure

objects in heart is $\bigcup_X \text{crt } W_X$ (Repr. of Jacobian algebra).

Associate crt. cohom. H_{crt} , maybe what we want?

(no Thom space here), after total integration, still circles & still get wall-crossings.

On objects of 3CY,

• str. of vanishing cycles

$R\Gamma$ (stack of objects, coeffs in this strat) \rightarrow critical cohomology.

3 CY categories: 2 classes

(1) A finite type dg algebra + 3CY structure

$\mathcal{C} =$ finite dimensional dg-modules

(non-cpt. CY, objects w/ cpt. support)

(2) A proper dg alg., 3CY.

$\mathcal{C} =$ perf A -mod.

Example: $X \rightarrow Y$ in (Y/X) (non-cpt)

Z (loc) subset in $Z_{\text{an. tp.}}$ $\mathcal{C} = \text{perf}_Z(\star)$
(cpt. sub) here / what's A? Take a strong generator, take Ext dgs.

\mathbb{Q}, W
polynomial

W formal power series.

In general, heredity alg. of ~~finite type~~ & fin. generators $\&$, formally smooth (kernel of mult. is ~~proj~~ projective??)

$\mathcal{A} \in \mathcal{A}/[\mathcal{A}, \mathcal{A}]$. (path alg. of quiv??)

fin. dim'l reps, \mathbb{C} crit. values.

map $K_0(C) \rightarrow \mathbb{Z}^n \xrightarrow{\text{stability}} \mathbb{C}$
 \mathbb{Z} central charge

Pick arbitrary perfect

$P_1, \dots, P_n \in \text{Perf}(A\text{-mod})$
 Σ fin. dim. $\rightarrow (\chi \text{ Rdim}(P_i, \Sigma))$

C^{∞} stack of finite type for any χ .

We need a constructible sheaf of vanishing cycles; does not come for free. moduli stack of all Σ 's

normalized dimension mod 2, det bundle / mod \mathbb{Z}^2 . \mathbb{Z} classes is $K^0(\text{Mod}, \mathbb{Z}/2) \sim H^1_{\text{et}}(\text{Mod}, \mathbb{Z}/2)$ like w_1, w_2 for File

Critical cohomology, much nicer global object

Suppose now X smooth proper (χ 3-fold) / \mathbb{C} .

Σ holom. vector C^{∞} rec. bundle.

\downarrow
 X

$\rightarrow A \in \Gamma(X, \Omega^{0,1} \otimes \text{End}(\Sigma))$ connection

$CS(A) = \int_X \text{Tr} \left(\frac{A \bar{\partial} A}{2} + \frac{A^3}{3} \right) \wedge \Omega^{3,0}$ same given $\int \omega$ for

get holom. form on ∞ , gauge gp. $= \text{Aut}(\Sigma)$

Gauge gp. doesn't preserve CS, one id. component over, on other components ω 's const, so $\int \omega$ (this functional) is preserved.

$\pi_0(\text{Aut } \Sigma) \rightarrow H_3(X; \mathbb{Z})$

value of CS shifted by $\int \Omega^{3,0}$ (class in $H_3(X; \mathbb{Z})$)

More generally, Σ \mathbb{Z} -graded C^{∞} bundle.

space of all $\{ \Sigma \text{ per connection} \}$ locally connected +

$\Gamma(X, \Omega^0 \otimes \text{End}(\Sigma))$

On \rightarrow get hol. closed 4-form

If we try to see sheaf of v.c. on this problem: finite-dim'l manifold.

if remove extra variables, reduce to fin. dimens. Requires answers on orientation question.

$\{ \text{super conn.} \}$ / connected component

$\downarrow H_3(X; \mathbb{Z})$ cover

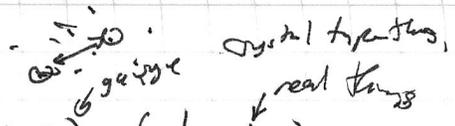
all connections

of id of $\text{Aut}(\Sigma)$ as \mathbb{Z} -graded bundle $\xrightarrow{CS} \mathbb{C}$ defined up to arch

A stability condition in the sense of Bridgeland
↓
stack of finite type.

critical values

$$(\int \Omega^{3,0}) (H_3(X, \mathbb{Z}) + \text{func } \& t)$$



At each critical value, expect orth. cuban
And for any straight line, additional operat on H_3 -
transcendental things, relating the crit values
e.g. - real gradient lines is viable modic connecting
one crit pt. to another, for potential

(Ref. written,
Chern simons
for 3-fold).

cross fingers
hope no
singularity @
∞.

so really solve somegn. on 7-dim \mathbb{R} a fold
(G2 moduli).

gradient flows in space of objects is 7-dim \mathbb{R} . (CY x \mathbb{R}) e.g.

Not part of the category theory → need to do G2 stuff
Get wall crossings of the type, this behave stability
conditions, moduli offers to disappear

So get some complicated new cohomology coming from CF theory.

If $X = CY^3$, $D(\text{oh}(X))$, get classical theory - + usual forms
before this G2 addition, which is non-algebraic-geometric.
• elliptic problem, analogue of Yang-Mills, self-dual...

These categories are like categories. Instead of cohomology of v.c., need
FS out of ∞ -dim \mathbb{R} objects.

- [FS cat has the wrong hodge str. 2, 2].

Q: Does this correspond to ~~something~~?

can have analogue of CS for Arnold, dilatation etc.

Don't really know how to translate --

maybe a kind of associative flows between crit pts on LC_3 's