Abouzaid II : A Generating Criterion for the Fukaya Cat.

Wrapped Fukaya category ⇝ Liouville manifold, i.e. Stein manifold, e.g. Affine variety.

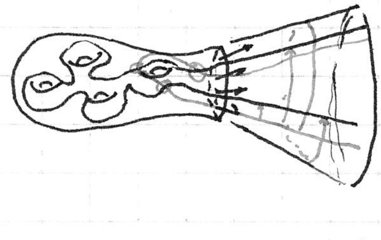
$M,\ \omega = d\lambda \leftarrow$ 1-form

with $\partial := \partial M$.

convexity at ∞ : $\exists$ Liouville vector field X_λ where $\boxed{\omega(X_\lambda, -) = \lambda}$

Condition: X_λ points outward along ∂.

Ex: If M is a Riemann surface, this can always be achieved.



- Objects : • Exact Lagrangians in M.
  $(\lambda|_L = df)$, + $f|_{\partial L}$ is locally constant.
- Morphisms : $CW^*(L_1, L_2)$ ↙ wrapped Floer complex
  $= \mathbb{Z}[L_1 \pitchfork L_2] \oplus \mathbb{Z}[\text{Reeb Chords from } L_1 \text{ to } L_2]$
  transverse intersection in interior. — In general of ∞ rank.

No chance of being CY in the proper sense.

Symplectic cohomology

The analogue of QH in the non-cpct. setting.

If M is Liouville, there is a well-defined notion of "quadratic at ∞" Hamiltonian function.

e.g. $M = T^*Q$, (p,q) coords., $H = |p|^2$.

Functor: ⇝ Floer homology, $SH^*(M)$

$SH^*(T^*Q) \overset{?}{\cong} H_{n-*}(\mathcal{L}Q)$

$\leftarrow SC^*(M)$ generated by ① crit pts. in the interior. (in $H^*(M)$) ② Orbits of the Reeb flow.

Circle actions are compatible ⇒ non-triviality of S^1-action.

Localization fails, SS w.r.t. S^1 action ~~[illegible]~~ doesn't degenerate

Given a collection of Lagrangians in Ob(W), when do they generate? (Remk: for $\mathbb{Z}$-grading, need orientation, spin)

$\mathcal{B} \subset \mathcal{W}$ full subcat.

Objects called L_{i}.

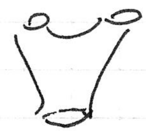
Answer:

$\exists$ a map $HH_*(\mathcal{W}, \mathcal{W}) \to SH^*(M)$.

If the identity of $SH^*(M)$ lies in the image of $HH_*(\mathcal{B}, \mathcal{B}) \to HH_*(\mathcal{W}, \mathcal{W}) \to SH^*(M)$

then $\mathcal{B}$ split-generates $\mathcal{W}$.

(Rmk: It's probably true that this is an iso.).

$SH^*(M)$ is a ring: product comes from .

But the identity comes from

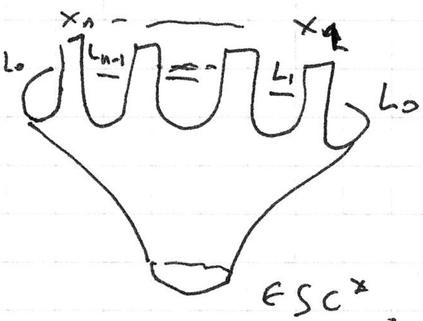
$$H^*(M) \xrightarrow{\text{ring map}} SH^*(M)$$

<u>what's the map?</u> (convention: input $\downarrow$ output)

Use the cyclic bar complex for HH_*.

Generator of CC_* $\rightsquigarrow$ $x_n \otimes \cdots \otimes x_1 \in CW^*(L_{n-1}, L_0) \otimes \cdots \otimes CW^*(L_0, L_1)$

map: (conformally infinite)



$\exists$ a map $SH^*(M) \to HH^*(\mathcal{W}, \mathcal{W})$

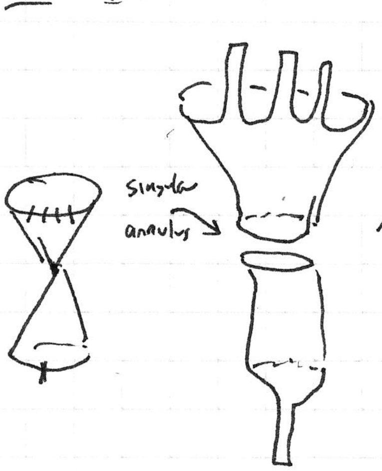
$\downarrow$ $HW^*(K)$ — end(K), K an object

ring map induced map is:

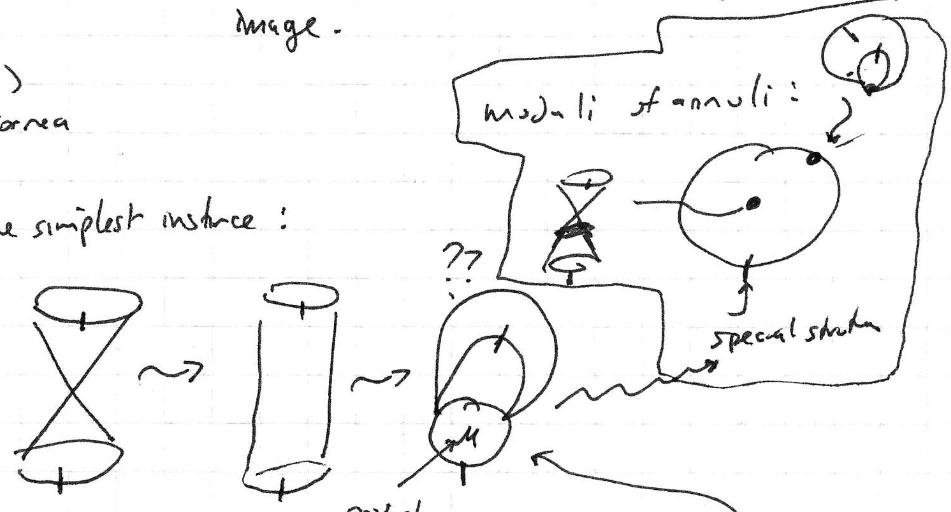
By assumption, get a map

$$HH_*(\mathcal{B}, \mathcal{B}) \longrightarrow HW^*(K,K)$$ for any K s.t. id_K is in the image.

<u>Idea</u>: (Related to ideas of • Fukaya • Biran – Cornea)



The simplest instance:



what's this in homological algebra?

$\mathcal{Y}^l(K)$ left Yoneda module over $\mathcal{B}$.
" $CW^*(K, L_i)$

$\mathcal{Y}^r(K) \rightsquigarrow CW^*(L_i, K)$.

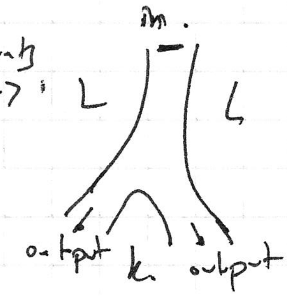
Claim: There exists a map of A_∞ bimodules

$$\Delta : \mathcal{B} \longrightarrow \mathcal{Y}^l(K) \otimes \mathcal{Y}^r(K)$$

diagonal bimodule

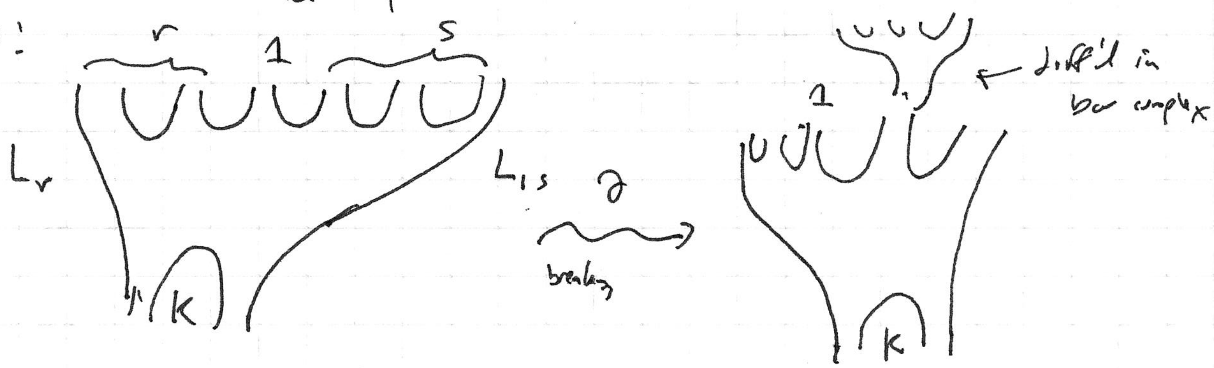
$\Delta \rightsquigarrow \Delta^{r|1|s} \quad \forall \quad r,s \geq 0.$

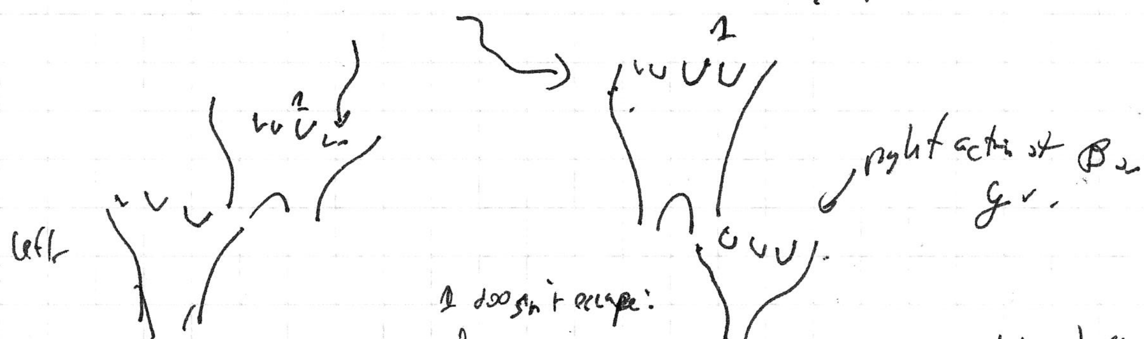
$\Delta^{0|1|0}$ counts $\rightsquigarrow$



$$HW^*(L,L) \longrightarrow HW^*(K,L) \otimes HW^*(L,K)$$

higher maps!





1 doesn't escape: depending on where 1 is, treat as right or left.

Given a map of A_∞ bimodules, get an ~~[illegible]~~ induced map on HH_*

$$\begin{tikzcd}
HH_*(\mathcal{B},\mathcal{B}) \arrow[r] \arrow[d, "HH_*(\Delta)"] & SH^*(M) \arrow[dd] \\
HH_*(\mathcal{B}, \mathcal{Y}^l(K) \otimes_k \mathcal{Y}^r(K)) \arrow[d, phantom, "\cong \text{ (some 2 cat. thing)}"] & \\
\mathcal{Y}^r(K) \otimes_{\mathcal{B}} \mathcal{Y}^l(K) \arrow[r, "HH_*(\Delta)(\sigma)"] & HW^*(K,K)
\end{tikzcd}$$

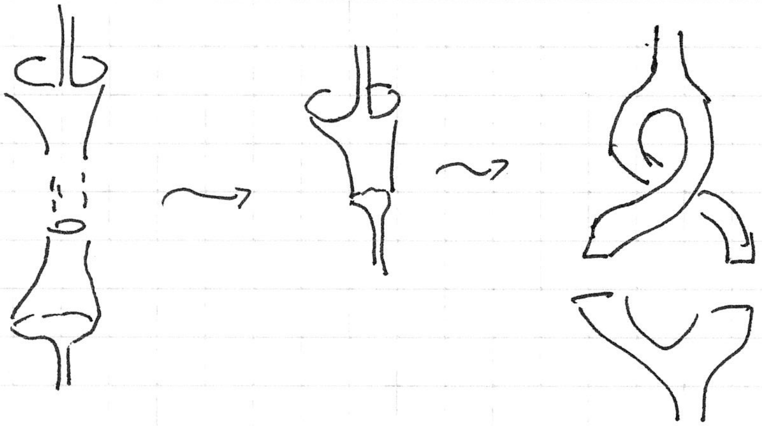
$\sigma \mapsto \mathrm{id}_M \mapsto \mathrm{id}_K$.

~~$HW^*(K,L)\otimes H$~~

$$HW^*(K,L) \otimes HW^*(L,K) \longrightarrow HW^*(K,K).$$

By Cardy relation $\Rightarrow$ diagram commutes

Picture for 1 input:



Look at diagram: $\Rightarrow$

$\exists\, f \in \mathrm{Hom}(K,L)$, $g \in \mathrm{Hom}(L,K)$

$f \circ g \simeq id_K \Rightarrow K$ is a summand of L

This is the one Lagrangian case. In general, might need to do this for many outputs, get a summand of the complex of Lag's.

This was all over $\mathbb{Z}$.

What about compact symplectic manifolds?

Work in progress with FOOO.

Several ways to proceed: • one way is just imitate this construction.

Another way:

Switch from $\mathbb{Z}$ to Novikov ring over $\mathbb{R}$. $\sim (\sum a_i t^{\lambda_i})$, $a_i \in \mathbb{R}$

Fukaya: The Fukaya category of a cpct. symplectic mfold is a ~~CY category~~ cyclic A_∞ category over $\Lambda_{\mathbb{R}}$. (technically, only done ~~for~~ for one Lagrangian, but never mind...).

A weak consequence is

$\mathcal{B} \subset \mathrm{Fuk}(M)$

lin. dual, Poincaré duality.

- $\mathcal{B}^\vee \simeq \mathcal{B}\,[\text{"}\dim\text{"}]$ as an A_∞ bimodule (cyclic CY says this iso has certain properties w.r.t. duality)

  (for this you don't need $\mathbb{R}$).

- $\mathcal{Y}_\ell^\vee(K) \simeq \mathcal{Y}_r(K)[n]$

~~Practice:~~

$\exists$ an evaluation map $\mathcal{Y}_\ell(K) \otimes_{\mathbb{K}} \mathcal{Y}_r(K) \xrightarrow{\mu} \mathcal{B}$

Dualize:

$$\mathcal{B}^\vee \longrightarrow \mathcal{Y}_r^\vee \otimes \mathcal{Y}_\ell^\vee$$

$$\cong \qquad\qquad \cong$$

$$\boxed{\mathcal{B}[n] \longrightarrow \mathcal{Y}_\ell \otimes \mathcal{Y}_r[2n]}$$

some shift --

This is the map we want. Define $\Delta = \mu^*[\pm n]$

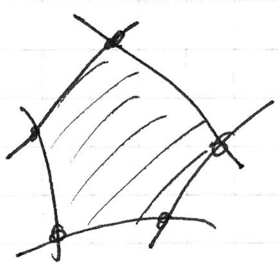
Back to diagram:

$$\begin{tikzcd}
HH_*(\mathcal{B},\mathcal{B}) \arrow[r, "\mathcal{P}", "\text{constructed by } FO^3"'] \arrow[d, "HH_*(\Delta)"] & QH^*(M) \arrow[dd] \\
HH_*(\mathcal{B}, \mathcal{Y}^\ell(K) \otimes_K \mathcal{Y}^r(K)) \arrow[d, phantom, "\cong"] & \\
\mathcal{Y}^r(K) \otimes_{\mathcal{B}} \mathcal{Y}^\ell(K) \arrow[r, "\mu"] & HF^*(K,K)
\end{tikzcd}$$

In the cpct case, the id $\in QH^*(M)$ is the fundamental class.

So the statement id lies in the image of P

$\Rightarrow \exists\ L_i, x_i \rightsquigarrow$ {Moduli of discs with boundary on L_i, corners on x_i, + 1 interior pt}



$$\downarrow ev$$

$$M$$

Criterion is: "sweep out space w/ hol. discs" (order 1)

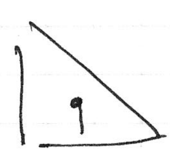
Kontsevich: (N.B. not actually HH_*, HH_* in some completed sense w.r.t. $\Lambda_{\mathbb{R}}$)

Consequence: $\mathrm{Fuk}(\mathbb{C}P^n) \cong D^b\,\mathrm{Sing}$ (Mirror)

$\cup$ Fuoo

{Subcategory gen. by Clifford tori with n+1 different local systems} [equiv] $\simeq$

Mirror has isolated non-deg. singularities n+1 crit points



These Clifford tori generate by looking at $HH_* \xrightarrow{\sim} QH^*$

How to check this generates $\mathrm{Fuk}(\mathbb{C}P^n)$? ~~These stuff~~

(already in literature that this is an iso.)