Kenji Fukaya: Cyclic symmetry and Numerical Invariant in Lag'n Floer Theory II.

CY case. | 0907.4219
(Part I = Fano case, Korea) | 0908.0148

Somehow mirror to hol. CS invariant.

C finite dim'l vector space OR $C = \Omega(L)$ DeRham complex, dim $L = n$

everything over

$< , >: C^k \otimes C^{n-k} \to \Lambda$ $\qquad \Lambda = \{ a_i T^{\lambda_i} \}$

perfect in case C f.d. $\qquad a_i \in K = \mathbb{R}$ (so we can use $\Omega(L)$).

In D.Rh case,

$\langle u, v \rangle := (-1)^{\deg u (\deg v + 1)} \int_L u \wedge v.$ ← sign designed so that $\langle u, v \rangle = (-1)^{1 + \deg' u \deg' v} \langle v, u \rangle$ where $\deg' u = \deg u + 1$ (anti-symmetric after deg. shift, Ref. "cyclic symplectic structure").

Def: Filtered A_∞-alg.

$m_{k,\beta}: C[1]^{\otimes k} \to C[1]$, $\deg = 1 - \mu(\beta)$, $\mu: G \to 2\mathbb{Z}$
like rel. $\pi_2(M, L)$

$m_k := \sum T^{E(\beta)} m_{k,\beta}$. $\qquad E: G \to \mathbb{R}_{\geq 0}$ proper (G is a monoid actually)

Def: cyclic A_∞ alg is $(C, \{m_k\}_{k=0}, < , >)$

s.t.

① $\sum_{\substack{k_1+k_2=k \\ i}} \pm m_{k_1}(x_1 \cdots m_{k_2}(x_i \cdots) \cdots) = 0$ (A_∞ rel'ns)

② $\langle m_k(x_1, \ldots, x_k), x_0 \rangle = (-1)^* \langle m_k(x_0, x_1, \ldots, x_{k-1}), x_k \rangle$

where $*$ is the natural koszul sign

$* = \deg' x_0 (\deg' x_1 + \cdots + \deg' x_k)$

(learned from Cho's paper).

Thm: (FOOO, + α) ($\alpha > \varepsilon$) $\quad L \subset M$ Lag. submfld, rel. spin

$(\Omega L \otimes \Lambda, < , >, \exists \{m_k\})$ cyclic filtered A_∞ algebra, well-defined up to pseudo-isotopy ← (might not technically be a well-defined notion.)

In what sense is this well-defined?

Two notions:

homotopy equivalence $f: (C, m, < , >) \to (C', m', < , >)$

$f_k: C[1]^{\otimes k} \to C'[1]$

s.t. ① $\sum m'_\ell (f_{k_1}(-) \cdots f_{k_\ell}(-))$

$= \sum f_{k_1}(x \cdots m_{k_2}(x -) - x)$ A_∞ homomorphism,

② $\sum_{k_1+k_2=k} \langle f_{k_1}(x_1, -, x_{k_1}), f_{k_2}(x_{k_1+1}, -, x_k)\rangle = \begin{cases} 0 & k \neq 2 \\ \langle x_1, x_2\rangle & k=2 \end{cases}$

(first found in a paper of Hajiwa?!)

③ f weak equivalence, i.e. $\cong$ on homology

But somehow, this is not what you get when you vary cx. structure, etc. Another notion is needed:

Def: $(C, \{m_k^t\}, \{c_k^t\}, \langle\ \rangle)$ is a pseudo-isotopy if:

$$C \hat{\otimes} \Omega^\cdot([0,1]) \ni x_i = a_i(t) + b_i(t)\,dt.$$

$m_k(x_1, -, x_k) = x + y\,dt$, $x = m_k^t(a_1(t) \cdots a_k(t))$.

satisfies A_∞ rel^n.

$y = \sum_i m_k^t(a_1, -, b_i(t) \cdots a_k) + c_k(a_1(t), -, a_k(t))$ $k \neq 1$.

OR

$y = m_1^t(b_1(t)) + c_1^t(a_1(t)) + \frac{da_1}{dt}$ $k=1$.

(In case $c=0$, m const., this is $\otimes$ of A_∞ alg, usual DeRham cplx. Allow t to vary:

m_k^t, c_k^t cyclically symmetric.

Problem: can't take inner product, b/c $[0,1]$ is non-cpct, so no P.D.

Def: $(C, \{m^0\}, \langle\ \rangle) \underset{\text{pseudo-iso}}{\sim} (C, \{m^1\}, \langle\ \rangle)$

$\iff \exists\, m^t, c^t$ as above

Lemma: If $(C, \{m^0\}, \langle\ \rangle) \overset{\text{p-iso.}}{\sim} (C, \{m^1\}, \langle\ \rangle)$

$\Rightarrow$ they are homotopy equivalent as cyclic A_∞ algebra.

Moreover, $\exists$ $f_t : (C, \{m^0\}, \langle\ \rangle) \longrightarrow (C, \{m^t\}, \langle\ \rangle)$

family of htpy equivalence

(can do this explicitly by summing over trees)

So p-isotopy is stronger than isotopy.

Point is, ~~resulting A_∞~~ varying cplx. structure $\rightsquigarrow$ pseudo-iso. c.s.

Now, consider $\underset{\underset{\text{dim.}}{\uparrow}}{n=3}$, $\eta : G \to \mathbb{Z}$ is 0. deg $m_{k,\beta} = 1$ after shift.

(e.g. slag, maslov is 0?).

Given:

$(C, \{m_k\}, \langle\ \rangle)$ associate $\Psi' : C^1 \to \Lambda_0$.

$$\Psi'(b) = \sum_{k=0}^{\infty} \frac{\langle m_k(b, \ldots, b), b\rangle}{k+1} \quad \text{superpotential}$$

(NOT same as Fano case).

$$\tilde{\mathcal{M}}(C) = \{ b \in C^1 \mid \sum_{k=0} m_k(b, \ldots, b) = 0 \}$$ Maurer-Cartan scheme

$\sim$ gauge equivalence: $\tilde{\mathcal{M}}(C)/_\sim = \mathcal{M}(C)$

In case of maslov 0, dim $=3$, $\mathcal{M}(C) \subset H^1(C, m_0^1) = H^1(L; \Lambda_0)$

Lemma: $b \in \tilde{\mathcal{M}}(C) \iff (\nabla \Psi')(b) = 0$.

(to prove this, need cyclic symmetry — crucial).

$(C, \{m^t\}, \{c_k^t\}, \langle\ ,\ \rangle)$ pseudo-isotopy induces

f^t (f_t ~~...~~ on earlier page) : $(C, m^0) \to (C, m^t)$ htpy equiv.

$f_*^t : \mathcal{M}(C, m^0) \xrightarrow{\sim} \mathcal{M}(C, m^0)$ Question: Does $\Psi_t'(f^t(b)) \overset{?}{=} \Psi'(b)$?

(i.e. is Ψ' a numerical invariant?)

NO.

How to see this?

Compute: $\frac{d}{dt} \Psi^t(f_*^t(b)) = \frac{d}{dt} \sum_k \frac{1}{k+1} \langle m_k^t(b_t, \ldots, b_t), b_t\rangle$

$b_t = f_*^t(b) = \sum_k f_k^t(b, \ldots, b)$

after some work,

$$= \sum_{k_1+k_2 \geq 1} \langle c_{k_1}^t(b_t, \ldots b_t), \underbrace{m_{k_2}^t(b_t, \ldots, b_t)}_{\text{things mostly cancel here, except bd. case:}} \rangle$$

$$= -\langle c_0^t(1), m_0^t(1) \rangle$$

To get a numerical inv't, need to get rid of this!

Def: $(C, \{m_k\}, \langle\ \rangle, m_{-1})$ is an inhomogeneous cyclic filtered A_∞ alg. iff:

$(C, \{m_k\}, \langle\ \rangle)$ is cyc. fil. A_∞ alg. and $m_{-1} \in \Lambda_+$.

• $(C, \{m_k^t\}, \{c_k^t\}, \langle\ \rangle, m_{-1}^t)$ is a pseudo-isotopy of inhom. cyclic filtered A_∞ alg.

$\Leftrightarrow$ • $(C, \{m_k^t\}, \{c_k^t\})$ pseudo-isot. of cyclic filtered A_∞ algebras and

• $\dfrac{d m_{-1}^t}{dt} = \langle c_0^t(1), m_0^t(1)\rangle$.

Define a corrected superpotential

$$\Psi : \mathcal{M}(C, \{m_k\}) \longrightarrow \Lambda_0$$

$$\Psi(b) = \sum_k \frac{1}{k+1} \langle m_k(b, \ldots, b), b\rangle + m_{-1}.$$

Lemma: If $(C, \{m^0\}, \langle\ \rangle, m_{-1}^0) \overset{p.\text{-}iso}{\sim} (C, \{m^1\}, \langle\ \rangle, m_{-1}^1)$
then $\Rightarrow \Psi(f_*(b)) = \Psi(b)$ on $\mathcal{M}(C, \{m^0\})$.

(This idea to include m_{-1} comes from D. Joyce, talked about by Seidel @ Oberwolfach)

Thm: $L \subset M$, $c_1(M) = 0$, $\dim_{\mathbb{C}} M$, L Lag'n submfld
rel. spin, $\mu_L := 0$ (Maslov index), J almost cplx str. w/ some condition ✯,

$\Rightarrow$ can define $(\Omega(L), \{m_k\}, \langle\ \rangle, m_{-1})$ inhom. cyc. fil. A_∞ alg.
It's well-defined up to pseudo-isotopy (still, we fixed J).

m_{-1} := contrib w/ no marked pts.

Condition ✯: $\mathcal{M}_1(\alpha, J) = \left\{ u : S^2 \to M \,/\, \begin{matrix} J\text{-holo.} \\ [u] = \alpha \end{matrix} \right\} \Big/ Aut(S^2, z_0)$

$\uparrow$ aut of S^2 fixing z_0. parabolic subgp of $SL(2, \mathbb{C})$.

$$\begin{array}{ccc} \mathcal{M}_1(\alpha,J) & & u \\ \downarrow ev & & \downarrow \\ M. & & u(z_0) \end{array}$$

$ev(\mathcal{M}_1(\alpha, J)) \cap L = \emptyset$ if $\alpha \neq 0$.

$\uparrow$ 2 dim. $\quad$ 3 dim.

If J is generic, can achieve this (some sort of genericity condition).

So get well-defined $\Psi^J : \mathcal{M}(L) \to \Lambda_0$ depends only on (M, L, J)
If $H_1(L) = 0 \Rightarrow \mathcal{M}(L) = pt$, $\Psi^J \in \Lambda_0$.. get an actual ~~num~~ invariant
(something like counts of discs bounding).

($\boxed{\text{What happens when we move } J?}$ Some sort of wall crossing —)

(fixing $J \sim$ fixing stability condition on the mirror

J_0, J_1 almost cx. str. satisfying ✩

$\mathcal{J} := \{J_t \mid t \in [0,1]\}$

$$\mathcal{M}_1(\alpha, \mathcal{J}) \overset{\text{3-dim.}}{=} \bigcup_t \mathcal{M}_1(\alpha, J_t)$$

$$\downarrow ev$$

$$M$$

$$\boxed{\begin{matrix} n(\alpha) = \#(L \cap ev(\mathcal{M}_1(\alpha, \mathcal{J}))) \\ \in \mathbb{Q}. \end{matrix}}$$

$$\begin{tikzcd} b \in \mathcal{M}(L, J_0) \arrow[r, "\sim"] \arrow[d, "\Psi_{J_0}"] & \mathcal{M}(L, J_1) \arrow[d, "\Psi_{J_1}"] \\ \Lambda_0 & \Lambda_0. \end{tikzcd}$$

b/c htpy type of A_∞-alg. ind. of J, so $\mathcal{M}(L)$ is too

Thm: (Wall-crossing)

$$\Psi_{J_0}(b) - \Psi_{J_1}(f_*(b)) = \sum_\alpha n(\alpha) T^{E(\alpha)}$$

Relations to 2 Earlier Works:

(1) J. Solomon (0606429). (Welschinger).:

$c_1(M) = 0 \quad \dim_{\mathbb{C}} M = 3$

$\tau : M \to M \quad \tau^2 = I,$

$\tau^* \tau = -J.$

$L = \{x \in M \mid \tau(x) = x\}$

Consider $\beta \in \pi_2(M, L)$, define $\mathcal{M}(\beta) := \left\{ u : (D^2, \partial D) \to (M, L) \;\middle|\; \begin{matrix} J\text{-hol.} \\ [u] = \beta \end{matrix} \right\} / \mathrm{Aut}\, D^2.$

$n_\beta = \# \mathcal{M}(\beta)$, then the invariant is

$$\sum T^{E(\beta)} n_\beta.$$

Found a way to make sense of this, ind. of choices.

F-O-O-O: $\tau : M \to M$ anti-hol. involution.

$L = \mathrm{Fix}\, \tau \qquad m_k : \Omega(L)^{\otimes k} \to \Omega(L)$

Under these conditions, have following symmetry:

$$m_{k,\beta}(x_1, \dots, x_k) = (-1)^* m_{k,\tau(\beta)}(x_k, \dots, x_1) \qquad \beta \in \pi_2(X, L)$$

$$* = \frac{1}{2}\mu(\beta) + k + 1 + \sum_{i<j} \deg' x_i \deg' x_j$$

(main thing is to study how τ changes orientations)

In particular, $m_0(\mathbb{1}) = -m_0(\mathbb{1}) = 0$.

(depends greatly on sign, bit delicate)

$\Rightarrow b = 0 \in \mathcal{M}(L)$, b/c elts. are sol'ns of $\cancel{m_0(\mathbb{1})} + \sum \cancel{m_i(b\ldots,b)} = 0$

(so only contribution from m_{-1}.

Then, Conjecture: Solomon's inv. $= \Psi_J(0)$.

(this is pretty close to a theorem).

Lemma: Suppose we have J_0, J_1 s.t. $\tau^* J_0 = -J_0$, $\tau^* J_1 = -J_1$ and J_t s.t. $\tau^* J_t = -J_t$.

Then, $\boxed{\Psi_{J_0}(0) = \Psi_{J_1}(0)}$

Another relation:

② M. Liu 0210257.

Setup: $L \subset M$, S^1 action on M, free on L.

$\beta \in \pi_2(M,L)$. $\dim \mathcal{M}(\beta) = 0$. Then $\# \mathcal{M}(\beta) \in \mathbb{Q}$ — maybe in $\mathbb{Z}$.

uses S^1-equivariant perturbations. S^1 is free on $\partial\mathcal{M}(\beta)$, which helps greatly.

Nice, b/c for individual β, well-defined.

May depend on S^1 action, so inv. of $S^1 \circlearrowright M$:

(Rmk: in this case can often use fixed pt. localization? to calculate $\#\mathcal{M}(\beta)$.)

Rel'n: $L \subset M$, S^1 action, free on L. $c_1(M) = 0$, $\mu_L : \pi_2(M,L) = 0$. then $\forall \beta$: $\dim \mathcal{M}(\beta) = 0$.

If $\#\mathcal{M}(\beta) \neq 0$, then $[\partial\beta] \sim k\,[S^1 \text{ orbit}]$

$[\gamma]$ S^1 orbit $\in H_1(L,\mathbb{Z})$. well-defined.

$[\gamma]$ defines a map $: H^1(L;\Lambda_0) \xrightarrow{x} \Lambda_0$.

$y := e^x : H^1(L;\Lambda_0) \longrightarrow \Lambda_0$.

Define:

$$\Phi: \sum_k \sum_{\partial\beta = k[\gamma]} \#\mathcal{M}(\beta)\, y^k\, T^{E(\beta)} : H^1(L;\Lambda_0) \longrightarrow \Lambda_0$$

Conj: ① $\{b \mid \nabla\Phi(b) = 0\} = \mathcal{M}(L)$ Maurer-Cartan scheme

② $\Psi_J = \Phi$ on $\mathcal{M}(L)$

J is S^1 invt. cplx. str.

~~To make sense~~ . Need to compare Lou's & Fooo's perturbations to prove this.

Keller – The Periodicity Conjecture via CY Categories :

mathematical physics

periodicity conjecture : applications to computations of central charge in CFTs & dilogarithm identities

Al. Zamolodchikov, 1991

proof based on homological algebra, study of certain 2-CY triangulated cats

philosophy of proof : $\boxed{\text{categorification}}$

ingredient 2: theory of cluster algebras (Fomin - Zelevinsky, 2002), relation between combinatorics & categorical setup

Plan:
1. The conjecture
2. The beginning of the proof : the categorification of finite reduced root systems
3. The end of the proof : homological periodicity.
4. Dessert: quiver - version of the conjecture.

Δ	h
A_n	$n+1$
D_n	$2n+2$
E_6	12
E_7	18
E_8	32

1. The conjecture

Δ, Δ' Dynkin diagrams (simply laced i.e. type A, D, E).

vertices : $I = \{1, \dots, n\}$, $I' = \{1, \dots, n'\}$

Cox. numbers h resp. h'. order of coxeter elt, see table!)

Incidence matrices: A resp. A' so $a_{ij} = \begin{cases} 1 & \exists\ i - j \\ 0 & \text{else.} \end{cases}$

Associated Y-system

variables : $Y_{i,i',t}$, $i \in I$, $i' \in I'$, $t \in \mathbb{Z}$.

equations :

$$Y_{i,i',t-1} \cdot Y_{i,i',t+1} = \frac{\prod_{j=1}^{n} (1 + Y_{j,i',t})^{a_{ij}}}{\prod_{j'=1}^{n'} (1 + Y^{-1}_{i,j',t})^{a'_{i'j'}}}$$

(i,j') — $(i,i') - (j,i')$

Conj: All solutions are periodic of period dividing $2\cdot(h+h')$.

[Gabriel - Happel : categorification of root systems]