Kuznetsov - Hochschild homology & cohomology of Admissible subcategories

scheme. (X smooth, projective).

for schemes $HH^{\cdot}(X) = \mathrm{Hom}^{\cdot}_{X\times X}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X)$.

$HH_{\cdot}(X) = H^{\cdot}(\cancel{X\times}X, \Delta_* \mathcal{O}_X \otimes^L \Delta_* \mathcal{O}_X) \cong \mathrm{Hom}^{\cdot}(\Delta_*\mathcal{O}_X, \Delta_* \omega_X[\dim X])$.

B - dg algebra, then $HH^{\cdot}(B) = \mathrm{Hom}_{B\otimes B^{op}}(B, B)$,

$HH_{\cdot}(B) = B \otimes^L_{B\otimes B^{o}} B$.

If $\mathcal{E}$ is a strong generator for $D^b(X)$, ~~then~~ $B = R\mathrm{Hom}(\mathcal{E}, \mathcal{E})$, then
$HH^{\cdot}(X) \cong HH^{\cdot}(B)$,
$HH_{\cdot}(X) \cong HH_{\cdot}(B)$.

Also, have nice geometric description:

$$HH_t(X) \cong \bigoplus_p H^{p+t}(X, \Omega_X^p)$$
$$HH^t(X) \cong \bigoplus_p H^{t-p}(X, \Lambda^p T_X).$$

Want to extend in nc-direction.

Def: A semi-orthogonal decomposition (SOD) of a triangulated cat. $\mathcal{T}$ is a sequence $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ of full triangulated subcats s.t.:

① $\mathrm{Hom}(\mathcal{A}_i, \mathcal{A}_j) = 0 \quad \forall i > j$

② $\forall T \in \mathcal{T} \quad \exists \quad 0 = T_m \to T_{m-1} \to \cdots \to T_2 \to T_0 = T$.
s.t. $\mathrm{Cone}(T_i \to T_{i-1}) \in \mathcal{A}_i$.

Nice -. If this is true, then such a filtration is unique & functorial.

Notation: $\mathcal{T} = \langle \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m \rangle$

Def: $\mathcal{A} \overset{\alpha}{\hookrightarrow} \mathcal{T}$ is called admissible if α has both left and right adjoint functors

If $\mathcal{A} \overset{\alpha}{\hookrightarrow} \mathcal{T}$ is admissible, then can extend to a SOD (usually in many ways),

e.g.
$$\mathcal{T} = \langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle = \langle \mathcal{A}^{\perp}, \mathcal{A} \rangle.$$

Also, if $\mathcal{T}$ smooth, proper, then each component of a SOD is admissible.

Let $\mathcal{A} \subset D^b(X)$ be an admissible subcat.
$\mathcal{E}$ - strong generator for $D^b(X)$.
$\mathcal{E}_{\mathcal{A}}$ - the component of $\mathcal{E}$ in $\mathcal{A}$.

$B_A = R\mathrm{Hom}(\mathcal{E}_A, \mathcal{E}_A)$.

Then define $HH^{\cdot}(\mathcal{A}) := HH^{\cdot}(B_A)$ (b/c $\mathcal{E}_A$ evidently a ~~[illegible]~~ strong gen. for $\mathcal{A}$).

$HH_{\cdot}(\mathcal{A}) := HH_{\cdot}(B_A)$

Functor $D^b(X) \longrightarrow D^b(X)$ <u>projection functor to $\mathcal{A}$</u>. (depends on choice of SOD which has $\mathcal{A}$ as component). fix one.

$\mathcal{E} \longmapsto \mathcal{E}_A$

<u>Lemma</u>: $\exists !\ P \in D^b(X \times X)$ s.t. the projection functor is iso to kernel functor

$$\Phi_P := p_{1*}(P \otimes p_2^{*}(\cdot))$$

$$\begin{tikzcd} & X \times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

(in general, not true for arbitrary).

Pf. (sketch) Say $D^b(X) = \langle \mathcal{A}_0, \ldots, \mathcal{A}_m \rangle$. (can see that, then

~~then~~ $D^b(X \times X) = \langle \mathcal{A}_{1x}, \ldots, \mathcal{A}_{mx} \rangle$ ($\mathcal{A}_{ix}$ = closure of objects gen. by $\mathcal{A}_i \otimes \mathcal{F}_K$ arbitrary).

$\rotatebox{90}{$\in$}$

$\Delta_* \mathcal{O}_X$ If P_i is the component of $\Delta_* \mathcal{O}_X$ in $\mathcal{A}_{ix}$,

~~then~~ $\Longrightarrow \Phi_{P_i}$ gives projection

<u>Thm</u>: $HH^{\cdot}(\mathcal{A}) \cong \mathrm{Hom}^{\cdot}(P, P)$ in $D^b(X \times X)$.

$HH_{\cdot}(\mathcal{A}) \cong H^{\cdot}(X \times X, P \overset{L}{\otimes} P^T)$ ← pull back under involution of $X \times X$.

<u>Pf</u>: $\mathcal{E}_{\mathcal{A}}$ $\mathcal{A} \cong D^{perf}(B_{\mathcal{A}})$

$$D^{perf}(B_A \otimes B_A^{op}) \xhookrightarrow{ff} D^b(X \times X)$$

$$B_A \longmapsto P.$$

For HH., check this functor compat. w/ tensor products in some sense

<u>Properties</u>:

1) <u>Functoriality of HH.</u> : $K \in D^b(X \times Y)$, $\mathcal{A} \subset D^b(X)$, $\mathcal{B} \subset D^b(Y)$ admissible, s.t. $\Phi_K(\mathcal{A}) \hookrightarrow \mathcal{B}$.

Then induces $\phi_K : HH_{\cdot}(\mathcal{A}) \longrightarrow HH_{\cdot}(\mathcal{B})$.

Can compute ~~very~~ explicitly in terms of K.

2) <u>$HH^{\cdot}$</u> (no functoriality w.r.t. arbitrary functors, but:) is <u>functorial w.r.t. equivalences</u>.

Now, say $\mathcal{A} \overset{\text{admissible}}{=} \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$. How are $HH_\bullet$, $HH^\bullet$ related?

- $HH_\bullet(\mathcal{A}) = HH_\bullet(\mathcal{A}_1) \oplus HH_\bullet(\mathcal{A}_2)$ (well-known for dg cats)

$$D^b(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$$

$$\begin{array}{ccc} P_1 & & P_m \\ \Phi_{P_1} & & \Phi_{P_m} \\ \phi_{P_1} & \text{———} & \phi_{P_m} \end{array}$$

are projectors on $HH_\bullet(X)$, and

$HH_\bullet(\mathcal{A}_i) = \mathrm{Im}\, \phi_{P_i} \subset HH_\bullet(X)$

Depends only on $\mathcal{A}_i$, not on SOD.

Q: Is this a decomposition as modules over $HH^\bullet$?? (Don't know) ~~[illegible]~~ ..

$\mathcal{A} = \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$

$HH^\bullet$: Let P be the kernel of projection functor corresponding to $\mathcal{A}$, $P_1, P_2 \leftrightarrow \mathcal{A}_1, \mathcal{A}_2$.

LES:

- $\cdots \to HH^t(\mathcal{A}) \to HH^t(\mathcal{A}_1) \oplus HH^t(\mathcal{A}_2) \to \overset{\bigstar}{\overbrace{\underbrace{\mathrm{Hom}^{t+1}(P_1, P_2)}}} \to HH^{t+1}(\mathcal{A}) \to \cdots$
- $\cdots \longrightarrow HH^t(\mathcal{A}) \longrightarrow HH^t(\mathcal{A}_1) \longrightarrow \mathrm{Hom}^{t+1}\underbrace{(P_2', P_2)} \to HH^{t+1}(\mathcal{A})$

where P_2' is the kernel of the pr. f. corresp. to

$$\mathcal{A} = \langle \overset{P_2'}{\mathcal{A}_2}, {}^{\perp}\mathcal{A}_2 \rangle.$$

Rmk: $\bigstar\,(\mathrm{Hom}^{t+1}(P_1, P_2)) \cong \mathrm{Hom}^t(\phi, \phi)$,

ϕ is the gluing functor, i.e. $= \alpha_2^{*\prime} \cdot \alpha_1$.

$$\mathcal{A}_1 \xhookrightarrow{\alpha_1} \mathcal{A} \xhookleftarrow{\alpha_2} \mathcal{A}_2.$$

Q: what is categorical interpr. of ?

Examples: ① Assume that $\mathcal{O}_X$ is exceptional. $\left(H^p(X, \mathcal{O}_X) = \begin{cases} k, & p = 0 \\ 0, & p \neq 0 \end{cases}\right)$

Then, $D^b(X) = \langle \mathcal{A}, \mathcal{O}_X \rangle$ Clearly, $HH_\bullet(\mathcal{A}) = HH_\bullet(X)/k$.

$\mathcal{A} = \mathcal{O}_X^{\perp}$, $\mathcal{O}_X \cong D^b(pt.)$.   (by summing).

For $HH^t(X) = \bigoplus_{p=0}^{\dim X} H^{t-p}(X, \Lambda^p T_X)$.

For $HH^t(\mathcal{O}_X^\perp) = \bigoplus_{p=0}^{\dim X-1} H^{t-p}(X, \Lambda^p T_X)$

Proof: Have $\mathcal{O}_X \boxtimes \mathcal{O}_X \to \Delta_* \mathcal{O}_X \to P$,

cone $\nearrow$ is projection functor.

Also, $HH^t(\mathcal{A}) = H^\cdot(X, \Delta^! P)$. So applying this, get:

$\Delta^!$.

$$\omega_X^{-1}[-\dim X] \xrightarrow{} \bigoplus \Lambda^p T_X[-p] \to \Delta^! P$$

$\uparrow$ certain Atiyah classes

class for $\mathcal{O}_X$ is trivial, so get an iso w/ top deg?

Assume $\langle E, \mathcal{O}_X \rangle$ - exc. pair in $D^b(X)$, E a v.b..

$$\mathcal{A} = \langle E, \mathcal{O}_X \rangle^\perp.$$

$$\cdots \to \bigoplus_{p=0}^{\dim X-1} H^{t-p}(X, \Lambda^p T_X) \to HH^t(\mathcal{A}) \to H^{t-\dim X+2}(X, E^\perp \otimes E \otimes \omega^{-1})$$

$$E^\perp = \ker(H^\cdot(E^\lambda) \to E^\lambda)$$

$$\sim \bigoplus_{p=0}^{\dim X-2} H^{t-p}(X, \Lambda^p T_X) \to HH^t(\mathcal{A}) \to H^{t-\dim X+2}(X, \mathcal{N}^\vee \otimes \omega_X^{-1}) \to$$

$$\mathcal{N}^\vee = \ker(E^\perp \otimes E \to \Omega_X).$$

If E is a line bdle, then one can check that is 0.

Last example: Let $f: X \to Y$ be a conic bundle.

Then $D^b(X) = \langle \underbrace{\mathcal{A}_X}, f^*(D^b(Y)) \rangle$

Q: How to compute $HH_\cdot$, $HH^\cdot$?

$D \overset{i}{\subset} Y$ the degen. locus

Have $\tilde{D} \xrightarrow{2:1} D$ non-ramified $\Rightarrow M \in \mathrm{Pic}^0 D$, s.t. $M^2 \cong \mathcal{O}_X$.

Then $HH_t(\mathcal{A}_X) = HH_t(Y) \oplus \bigoplus_{p=0}^{\dim X-2} H^{p+t}(D, \Omega_D^p \otimes M)$

$$HH^t(\mathcal{A}_X) = \bigoplus_{p=0}^{\dim Y} H^{t-p}(Y, \ker(\Lambda^p T_Y \to i_*(\mathcal{N} \otimes \Lambda^{p-1} T_D)))$$

Non-vanishing conjecture: Let X be smooth, projective variety, and $\mathcal{A} \subset D^b(X)$ admissible subcat. Then if $HH_\cdot(\mathcal{A}) = 0 \Rightarrow \mathcal{A} = 0$.

Rmks: If $\mathcal{A}$ is a CY subcat, then it is true.. (b/c $HH_\cdot(\mathcal{A}) \cong HH^\cdot(\mathcal{A})$ [shift], but certainly $HH^\cdot$ (non-trivial) is always non-trivial, identity iso).

Cor. 1: If $\mathcal{A}_1, \dots, \mathcal{A}_n$ semi-o. collection in $D^b(X)$ and $\bigoplus HH_\cdot(\mathcal{A}_i) = HH_\cdot(X)$. Then $D^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ a S.O.D.

Useful, b/c it tells us when we're done constructing our collection.

Cor 2: Any increasing chain

$\mathcal{A}_1 \subset \cdots \subset D^b(X)$ of admissible subcats. stabilizes.

(some noetherian-like property)

(easy b/c $HH_*(D^b(X))$ is fin. dim.).

Abouzaid III: $M = T^*Q$ ← smooth cpct mfold

Thm: $\exists$ ⊛ a fully faithful embedding $\mathcal{W}(T^*Q) \to \mathrm{mod}(C_{-*}(\Omega_q Q))$ whose image agrees with the triangulated closure of the free module.

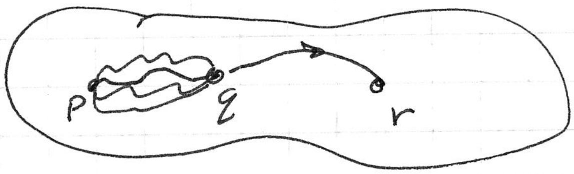
(the functor always exists, proof of fully faithful assumes $\tilde{Q}$ ← univ. cover has finite htpy type)

$\Rightarrow T^*_q Q$ cotangent fibre generates the Fukaya category, and every exact Lag'n in T^*Q has "filtration" by cotangent fibres.

Define: $\mathcal{P}(Q) = \begin{cases} \underline{\text{ob}}: q \in Q \\ \underline{\text{Mor}}: \mathrm{Hom}(p,q) = C_{-*}(\Omega_{p,q} Q) \\ \underline{\text{composition}}: \text{concatenation} \end{cases}$

Morphisms = dga if you use Moore paths & cubical chains

↑ keep track of length, as opposed to $[0,1]$ ⇝ gives associativity on the nose.



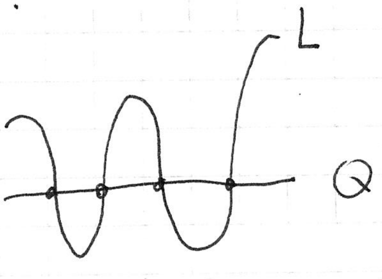
Whenever $Q \xrightarrow[\text{exact Lag'n}]{} M$ ← Liouville mfld, $\exists$ a functor $\mathcal{W}(M) \to Tw(\mathcal{P}(Q))$

Twisted Complexes:

$(X_i, D = (\delta_{ij}))$ $\delta_{ij} = 0$ if $i \geq j$.

$\partial D \pm D^2 = 0$ $(\deg \delta_{ij} = 1)$ e.g.

$$\begin{tikzcd} X_1 \arrow[r] \arrow[rr, bend left] & X_2 \arrow[r] & X_3 \end{tikzcd}$$

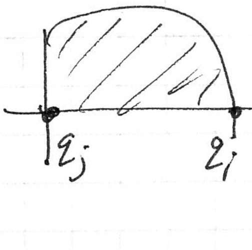


"X_i" ⇝ $q_i \in Q \cap L$

- If graded, shift by Maslov index of q_i
- Order by "action".

Key Fact:

If $A(q_i) < A(q_j)$, then the moduli space of strips which converge ~~at~~ $-\infty$ to q_j and $+\infty$ to q_i is empty



e.g. this strip exists, but not vice versa