Cor 2: Any increasing chain
$\mathcal{A}_1 \subset \cdots \subset D^b(X)$ of admissible subcats. stabilizes.
(some noetherian-like property)
(easy b/c $HH_*(D^b(X))$ is fin. dim.).

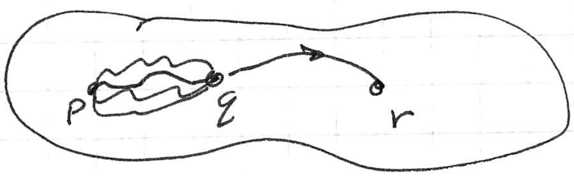
Abouzaid III: $M = T^*Q$ ← smooth cpct m'fold

Thm: $\exists$ ⊛ a fully faithful embedding $\mathcal{W}(T^*Q) \to \mathrm{mod}(C_{-*}(\Omega_q Q))$
whose image agrees with the triangulated closure of the free module.
(the functor always exists, proof of fully faithful assumes $\tilde{Q}$ ← univ. cover has finite htpy type)

$\Rightarrow T_q^*Q$ cotangent fibre generates the Fukaya category.
and every exact Lag'n in T^*Q has "filtration" by cotangent fibres.

Define: $\mathcal{P}(Q) = \begin{cases} \underline{\text{ob}}: q \in Q \\ \underline{\text{Mor}}: \mathrm{Hom}(p,q) = C_{-*}(\Omega_{p,q} Q) \\ \underline{\text{composition}}: \text{concatenation} \end{cases}$

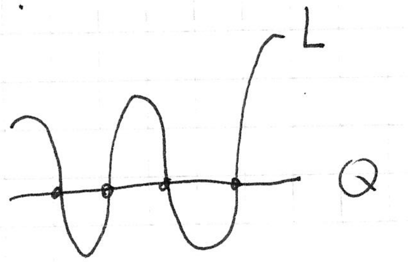
Morphisms = dga if you use Moore paths & cubical chains
↑ keep track of length, as opposed to $[0,1]$ ⇝ gives associativity in the nose.



Whenever $Q \xhookrightarrow[\text{exact Lag'n}]{} M$ ← Liouville mfld, $\exists$ a functor $\mathcal{W}(M) \to Tw(\mathcal{P}(Q))$

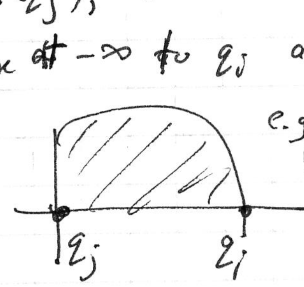
Twisted Complexes:
$(X_i, D = (\delta_{ij}))$ $\delta_{ij} = 0$ if $i \geq j$.

$\partial D \pm D^2 = 0$ $(\deg \delta_{ij} = 1)$ e.g. $X_1 \to X_2 \to X_3$



"x_i" ⇝ $q_i \in Q \cap L$
- If graded, shift by Maslov index of q_i
- Order by "action"

Key Fact:
If $A(q_i) < A(q_j)$, then the moduli space of strips which converge at $-\infty$ to q_j and $+\infty$ to q_i is empty

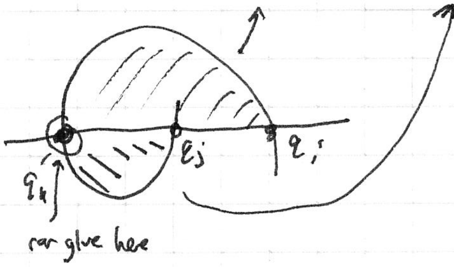


e.g. this strip exists, but not vice versa

For each pair, consider $\bar{\mathcal{M}}(q_i, q_j) = \{$ compactified mod. space of hol. strips $\}$.

This is a manifold with boundary, and the boundary is covered by codim 1 strata which are the images of

$$\bar{\mathcal{M}}(q_i, q_k) \times \bar{\mathcal{M}}(q_k, q_j) \to \bar{\mathcal{M}}(q_i, q_j)$$



$\bar{\mathcal{M}}$ admits a fundamental chain. (very non-unique).

We choose such chains compatibly, so that:

$$(*)\ \partial[\bar{\mathcal{M}}(q_i, q_j)] = \sum \pm [\bar{\mathcal{M}}(q_i, q_k)] \times [\bar{\mathcal{M}}(q_k, q_j)].$$

$\exists$ an evaluation map

$$\bar{\mathcal{M}}(q_i, q_j) \xrightarrow{ev} \mathcal{P}(q_i, q_j$$

Define: $\delta_{ij} = ev_*[\bar{\mathcal{M}}(q_i, q_j)]$.

$(*) \Rightarrow$ M-C equation for a twisted complex

Filtration comes from intersection w/ 0-section (one fibre for each intersection).

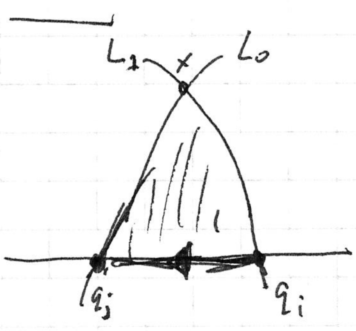
$$\text{Given } L_0, L_1 \rightsquigarrow CW^*(L_0, L_1) \to \mathrm{Hom}(\mathcal{F}(L_0), \mathcal{F}(L_1)). \quad (\mathcal{F} = \text{"functor"}).$$

("generated by $L_1 \cap L_2$")

Recall: $\mathrm{Hom}((T_1, \delta^1_{ij}), (T_2, \delta^2_{ij})) = \bigoplus_{i,j} \mathrm{Hom}(X^1_i, X^2_j)$,

$T_1 \to X^1_i$, $T_2 \to X^2_j$

Diff: $= \partial \pm - \circ D^1 \pm D^2 \circ -$

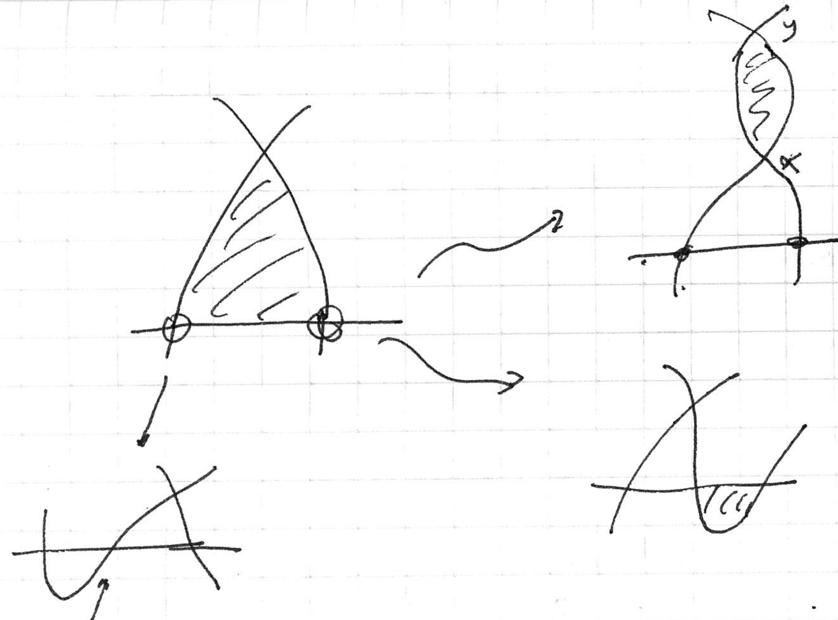


$\mathcal{M}(q_j, q_i, X) =$ hol. maps from a "triangle" to M with corners at x, q_i, q_j

$$\begin{array}{c} \mathcal{M}(q_j, q_i, X) \\ \big\downarrow ev \\ \Omega_{q_i, q_j} Q \end{array}$$

Choose $[\mathcal{M}(q_j, q_i, x)]$ & define. $\mathcal{F}^1(X) = \bigoplus_{q_i, q_j} ev_*([\mathcal{M}])$

Analyze breaking:

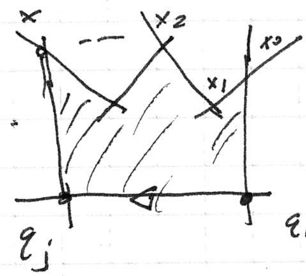


$\mathcal{F}'\circ\mu'$ ~ differential in CW^*

$\mathcal{F}'\circ D'$ ← Differential in $\mathcal{F}(L_i)$

$D^2 \circ \mathcal{F}'$

In general, for higher ~~terms~~, look at:



This is where the stupid htopy theoretic condition appears. Abbondandolo-Schwarz prove ∃ a map

Go back to the case $M = T^*Q$:

① Proving that $\mathcal{F}$ induces an A_∞ quasi-iso. from $CW^*(T_q^*Q, T_q^*Q) \to C_{-*}(\Omega_q Q)$

② Prove that the fibre generates.

$C_{-*}(\Omega_q) \xrightarrow{AS} CW^*(T_q^*Q)$

inducing an iso. on homology, (but not extended to A_∞ map.)

Prove map we used is a one-sided inverse to AS.

If you impose stupid conditions, the vector spaces are graded by degree & htopy class, so everything is fin. dim'l, etc.

Proof of generation:

Recall: $HH_*(CW^*(T_q^*Q)) \to SH^*(T^*Q)$ hits the identity, then T_q^*Q split-generates..
(then ε-homological alg. gets generation.)

Setup a diagram:

$$\begin{array}{ccc} HH_*(CW^*(T_q^*Q)) & \xrightarrow[\text{yesterday}]{\text{degn?}} & SH^*(T^*Q) \\ \text{Part ①}\ \wr\downarrow & & \downarrow \Big\}\ \text{almost an iso.} \\ HH_*(C_{-*}(\Omega_q Q)) & \xrightarrow{\quad} & H_{-*}(\mathcal{L}Q) \end{array}$$

degn? ← Iso by Goodwillie.

Can do enough to at least show that 1 is in the image.

If you choose a clever model for chains,

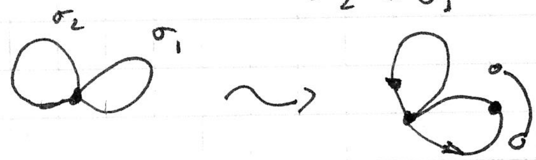
$$HH_* \rightsquigarrow \underset{\text{cyclic bar cplx}}{CC_*} \rightsquigarrow \bigoplus CC_*^{\overset{N}{\uparrow}} (C_{-*}\Omega_q Q)$$

length of words

$$CC^1_* \rightsquigarrow C_{-*}(\Omega_q Q) \xhookrightarrow{\text{inclusion}} C_{-*}(\mathcal{L}Q)$$

$$CC^2_* \rightsquigarrow C_{-*}(\Omega_q Q) \otimes C_{-*}(\Omega_q Q)[1]$$

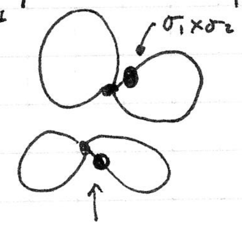
$$\sigma_2 \otimes \sigma_1 \longrightarrow \sigma_2 \boxtimes \sigma_1$$



Move basept around the circle coming from σ_1

Chain map?

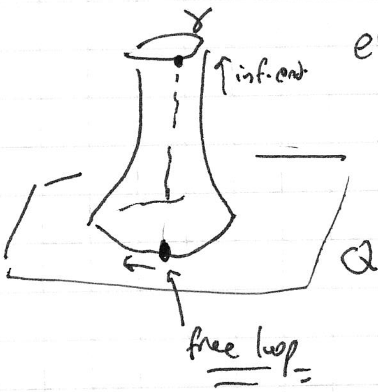
$$\partial(\sigma_2 \otimes \sigma_1) = \underbrace{\partial\sigma_2 \otimes \sigma_1 + \sigma_2 \otimes \partial\sigma_1}_{\text{easy to find.}} + \sigma_2 \times \sigma_1 + \sigma_1 \times \sigma_2 .$$



$$CC_*^{\geq 3} \rightsquigarrow 0.$$

(really, should be putting homotopies everywhere).

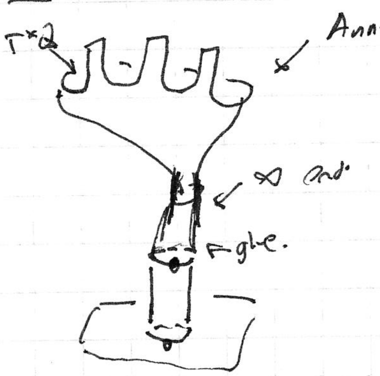
Gen. of $SC^*(T^*Q) \rightarrow$ "Reeb orbits" with starting point.



$$ev_*[\mathcal{M}(\gamma)] \subset C_{-*}(\mathcal{L}Q)$$

Check: diagram commutes!

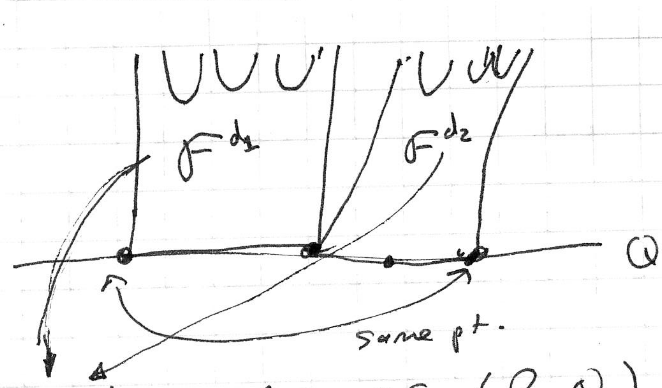
Yesterday's map:



Annulus w/ ∞-modulus

Glue $\Rightarrow$ Annulus w/ finite modulus (some sort of real condition).

On the "other" side of the moduli space, :



terms of A_∞ map $CW^* \to C_{-*}(\Omega_q Q)$,

~~diagram commutes,~~ b/c
Recall:

$\mathcal{F} : \mathcal{A} \to \mathcal{B}$, then induced map on cyclic bar complex is:

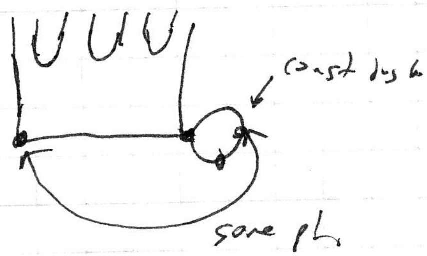
$$a_n \otimes \cdots \otimes a_1 \longrightarrow \sum \mathcal{F}^{d_1}(a_{s_k}, \ldots, a_1, a_n, \ldots, a_{s_1}) \otimes \cdots \otimes \mathcal{F}^{d_k}(a_{s_{k-1}}, \ldots a_{s_k - 1})$$

So this map is exactly

$$\begin{array}{l} HH_*(CW^*(T_q^*Q)) \\ \downarrow \qquad\qquad \searrow \\ HH_*(C_{-*}(\Omega_q Q)) \;\star \quad H_{-*}(\mathcal{L}Q) \end{array}$$

b/c $\star$ kills maps of order ≥ 3

How does Length 1 arise? $\longrightarrow$



$\Rightarrow$ T_q^*Q split generates.
But we have a functor to $Tw(\mathcal{P}(Q)) \cong$ triangulated closure of T_q^*Q,
where a point (non-split) generates, ✓

Q: Can a summand of an iterated cone also be represented as a cone?
A: (Open?) For a $K(\pi, 1)$, becomes a question about K-theory of grp ring, so probably false.

Yes for X s.c., b/c $C^*(X)$ has a unique simple module
(prove by taking min model, A_∞ all comes d_1.)

$C_*(\Omega X) = B$ $\quad \mathrm{mod}(B) \longrightarrow \mathrm{mod}(C^*(X))$ fully faithful only if X s.c.
(other way true)