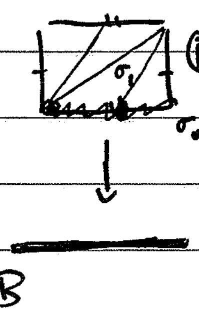
M. Gross I

Two daydreams

① (Andrei Tyurin, '99) : HMS predicts

Lag'n sections of s. lag. fibrations are mirror to line bundles.

ex.



$\sigma_0 \longleftrightarrow \mathcal{O}_E$

$\sigma_1 \longleftrightarrow \mathcal{O}_E(p)$

$\sigma_2 \longleftrightarrow \mathcal{O}_E(2p)$

Expect $\underset{d>0}{HF^*(\sigma_0, \sigma_d)} \cong H^0(\mathcal{O}_E(dp))$

Given by

$$\bigoplus_{q \in \sigma_0 \cap \sigma_d} \Lambda[q]$$

Expect a nice basis for $H^0(\mathcal{L}^{\otimes d})$ related to the intersection points of σ_0 and σ_d.

"Theta functions for Calabi-Yau manifolds."

② Find a general construction of mirrors.

Suppose we are given a maximally unipotent degeneration $\mathcal{Y}_0 \subseteq \mathcal{Y}$ of C-Y manifolds.

$$\begin{array}{ccc} \mathcal{Y}_0 & \subseteq & \mathcal{Y} \\ \downarrow & & \downarrow \\ 0 & \in & D \end{array} \quad \text{disk}$$

Can we find a mirror for this degeneration?

(monodromy $T: H^n(\mathcal{Y}_t, \mathbb{C})\circlearrowleft$, $(T-I)^{n+1} = 0$, $(T-I)^n \neq 0$.)

in general always true that $(T^m - I)^{n+1}$, get rid of by base change?

Program with Siebert applies if $\mathcal{Y} \to D$ is a nice toric degeneration.
(e.g. complete intersections in toric varieties).

<u>This talk:</u> • Construction of theta functions
(works in any dimension, but will focus on two dim's, where it works more generally).
• Solve ② in two dimensions.
(joint w/ Paul Hacking & Sean Keel)
• HMS? (joint with Abouzaid + Siebert.)

Start with $\begin{array}{c}\mathcal{Y}\\ \downarrow\\ D\end{array}$ max. unipotent degen. of K3 surfaces, normal crossings, relatively minimal.

$A_1(\mathcal{Y}/D)$ = abelian group generated by curves in $\mathcal{Y}$ mapping to points in D modulo numerical equivalence (fin. rank)

$NE(\mathcal{Y}/D) \subseteq P \subseteq A_1(\mathcal{Y}/D)$ monoid.

cone ~~gen'd~~ generated by effective curves.

choose P a toric monoid (intersection of rat'l polyhedral cone w/ lattice, $\sigma \cap A_1(\mathcal{Y}/D)$.

for some strictly convex rational polyhedral cone $\sigma \subseteq A_1 \otimes_{\mathbb{Z}} \mathbb{R}$.

Let $J = P \setminus \{0\}$. Can think of J as a monomial ideal in $k[P]$. Let $\widehat{k[P]}$ be the completion of $k[P]$ w.r.t. J.

Goal: Construct a mirror family over $\mathrm{Spec}\ \widehat{k[P]}$.

like Kähler moduli space.

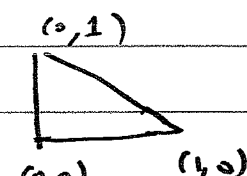
Construct an affine manifold with singularities, B, from $\mathcal{Y} \to D$.
B is the dual intersection complex of $\mathcal{Y}_0$, i.e. vertices of B correspond to irreducible components of $\mathcal{Y}_0$.

$v \longleftrightarrow Y_v$

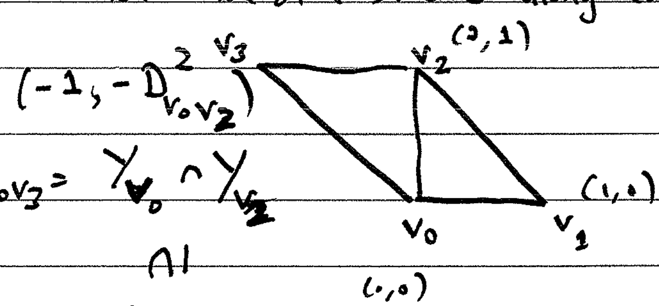
$\langle v_0, \ldots, v_n \rangle$ is a simplex of B if $Y_{v_0} \cap \cdots \cap Y_{v_n} \neq \emptyset$.

B is a simplicial complex homeomorphic to S^2.
(rel. minimal max. unipotent implies this)

Each 2-dim'l cell can be given an affine structure on its interior via an identification with the standard simplex.



We extend the affine structure along edges as follows:



$D_{v_0v_2} = Y_{v_0} \cap Y_{v_2}$

$\cap|$

Y_{v_0}

Looks asymmetric, but it works out: $(D^2_{v_0v_2} + D^2_{v_2v_0} = -2$ implies well-defined).

singularities are now on vertices.
This affine structure extends across vertex v iff $(Y_v, \mathrm{Sing}(\mathcal{Y}_0) \cap Y_v)$ is a toric pair.

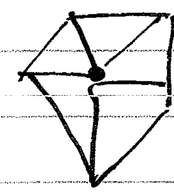
Let $\Delta \subseteq B$ be the set of vertices; we now have an affine structure on $B_0 = B \setminus \Delta$.

e.g.: $\exists\ Y_0$ which is a union of 12 del Pezzo surfaces of degree 5, glued like a dodecahedron.

B is an icosahedron with edges flattened.

B can be thought of as the intersection complex for the central fibre of our family over the closed point of Spec $k[P]$.

This will be a union of $\mathbb{P}^2$'s.



Deforming central fibre

Let $\Lambda \subseteq J_{B_0}$ be the sheaf of flat integral vector fields. (locally generated by $\partial/\partial y_1, \ldots, \partial/\partial y_n$, $n=2$) where $y_1, \ldots, y_n$ integral affine coordinates.

For ρ an edge of B, $x \in \mathrm{Int}(\rho)$, and choose a piecewise linear Λ function $\varphi_\rho : \Lambda_x \otimes \mathbb{R} \to \rho^{gp} \otimes \mathbb{R}$.

$\Lambda_x \otimes \mathbb{R}$

Λ_ρ tangent space to ρ

m $m \in \Lambda_x$ maps to a primitive generator of Λ_x / Λ_ρ.

lives in monoid.

The bend of φ_ρ is $\varphi_\rho(m) + \varphi_\rho(-m) = [D_\rho]$

$\rho = \langle v, w \rangle$ $D_\rho = D_{vw} = Y_v \cap Y_w$

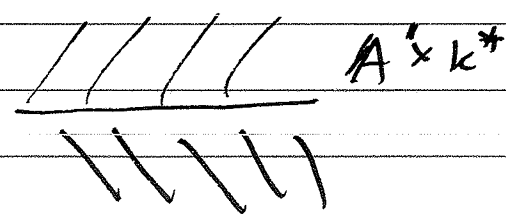
Let $P_\rho = \{(m,p) \mid p - \varphi_\rho(m) \in P\} \subseteq \Lambda_x \oplus P^{gp}$.

Obviously a P-action.

Get:

$$k[P_\rho] \leftarrow k[P]$$
$$\cong$$
$$\frac{k[X_+, X_-, X_0^{\pm 1}][P]}{(X_+ X_- - z^{[D_\rho]})}$$

rate of smoothing, in certain direction.



Fix $I \subseteq P$ a monomial ideal, with $\sqrt{I} = J$.

$\left(k[P] := \bigoplus_{p \in P} k \cdot z^p \right)$

Define $U_{\rho, I} = \operatorname{Spec} k[P_\rho] \otimes_{k[P]} k[P]/I$.

If σ is a 2-cell with edges ρ_1, ρ_2, we have natural inclusions

$$U_{\rho_1, I} \supseteq U_{\sigma, I} \subseteq U_{\rho, I}.$$
$$U_{\sigma,I} = \operatorname{Spec} k[\Lambda_\sigma] \otimes_k k[P]/I.$$

Gluing over these inclusions gives a flat deformation $X_I^\circ \to \operatorname{Spec} k[P]/I$.

(No chance it compactifies. Modify gluings so that it does.)