Kontsevich I

2nd & 3rd talks: issues w/ gamma fns,

1st: introduction to subject

real integrable system: $(X^{2d},\omega) \downarrow^{\pi} B$ proper, generic fiber of π, $\pi^{-1}(q)$ Lag'n torus

codim $B^{sing} \geq 2$.

(bit hard in practice but sometimes achievable)

non-precise condition: "convexity at ∞ in base" (fibers proper)

canonically $\Longrightarrow$ $(X,\omega)^{\vee} = X^{\vee}$

analytic, C-Y of dimension $= d$ over Novikov field.

$$\Lambda := \sum_{c_i \in \mathbb{Q}} c_i T^{\lambda_i} \quad \lambda_i \in \mathbb{R}, \ \lambda_i \to +\infty$$

$X^{\vee} :=$ moduli space of objects in $\mathcal{F}(X,\omega)$

"fibers of π with $U(1)$-local system"

seems like a perfectly reasonable idea:

for degen. cx. str (Fukaya), hol. curves are lines in base, & circle in fibers

not $c_1 = 0$ a priori (could be $\mathbb{Z}/2g$-graded ...)

Choose $\mathbb{R}$ metric g on base, get wall-crossings, etc.

A different intrinsic definition:

Namely, on $X^\vee$, have a volume form $\Omega^{d,0}_{X^\vee}$

$$\underset{\text{open}}{U} \subset X^\vee \quad (\text{remove } \hat{\pi}^{-1}(B^{\text{sing}}))$$

semi-flat

$$U \underset{\text{homeo}}{\approx} \{\text{fibers of } \pi \text{ with } U(1)\text{-local system}\} =: (X^\vee)_{sf}$$

$$\Omega^{d,0}_{sf}$$

(need oriented base for this)

Also, $[\Omega^{d,0}_{X^\vee}] = [\Omega^{d,0}_{X^\vee_{sf}}] \in H^d(U)$

(In a sense, Tian-Todorov thm, says c.s. cohomology uniquely determined by open part)

No instanton corrections, (modifications don't change homology class.

complex integrable system

$$\dim_{\mathbb{C}} X = 2n, \quad n = 2d, \quad \omega^{2,0}$$

$$\operatorname{codim}_{\mathbb{R}} B^{\text{sg}} \geq 2 \text{ automatically}$$

$\forall \zeta \in \mathbb{C}^\times$

$$X, \ \mathrm{Re}(\zeta^{-1}\omega^{2,0})$$

$$\dashrightarrow X^\vee_\zeta$$

In this case, on $X^\vee_\zeta$, symp. form $\Omega^{2,0}_{X^\vee_\zeta}$, with properties the same

$$[\Omega^{2,0}_{X^\vee_\zeta}] = [\Omega^{2,0}_{X^\vee_{sf,\zeta}}] \in H^d.$$

Vol. form is some power of sympl. form.

Walls $\subset \underbrace{B \times \mathbb{C}^* / \mathbb{R}_+}_{S^1}$ as move ξ.

cohomology class of form depends hol. on parameter
(once one adds a B-field, twist),
all B-fields $\underline{\xi^{-1}\omega^{2,0}} + \bar{\xi}\,\bar{\omega}^{2,0}$.

cx. int. system of Seiberg-Witten type:
$[\omega_X^{2,0}] = 0$. (non-compact variety

Then something nice goes on:
$\rightarrow X^\vee_\xi$ is a local system over $\xi \in \mathbb{C}^\times$
of affine alg. varieties $/\mathbb{Z}$
"cluster varieties"

(why affine alg. variety?)

flat coordinates {
$(u,\xi) \notin$ wall : $\mathbb{Z}[[x_1, \dots, x_{2n}]][x_1^{-1}, \dots, x_{2n}^{-1}] \supset \mathcal{O}(X^\vee_\xi)$
finitely gen'd.

When consider X w/ real sympl. form, will have real symp. manifold
w/ exact form,
hopefully Liouville / ~~exact~~ Weinstein, so Fuk. cat. is calculable
so Fukaya category = finite modules over ~~these~~ some algebra.
wall crossing gives certain subrings

$\forall\, u \in B - B^{sing}$

get $Z_u : \Gamma_u = H_1(\pi^{-1}(u), \mathbb{Z}) \to \mathbb{C}$ charge.

$\omega^{2,0} = d\alpha^1$. (non-hol) depends on choice of one-form.

$Z_u : \gamma \longmapsto \int_\gamma \alpha^1$

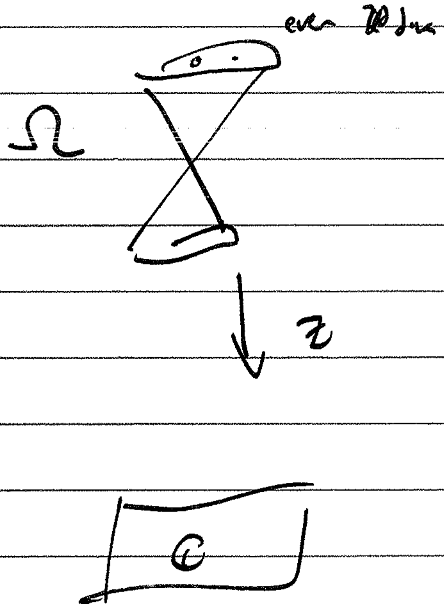
$(u, \zeta) \notin$ walls, # BPS states $\Omega_{u}(\gamma) \in \mathbb{Z}$

Given $u \in B - B^{sing}$, $\zeta \in \mathbb{C}^\times$

$(u, \zeta) \notin$ walls

$\forall\, \gamma$, $\Omega_u(\gamma) \in \mathbb{Z}$

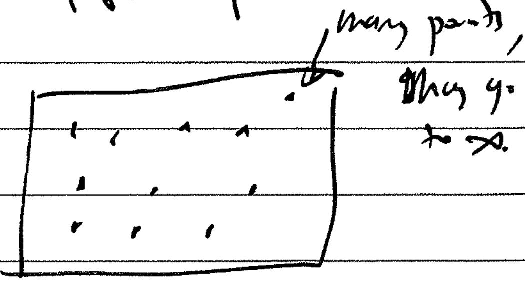
$\rightsquigarrow$ reconstruct $X_\zeta^\vee$?



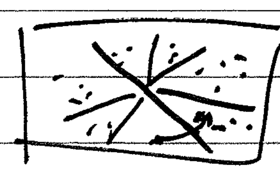
$Z : \operatorname{supp} \Omega \to \mathbb{C}$

image discrete subset,

So proper map.



Divide into several sectors

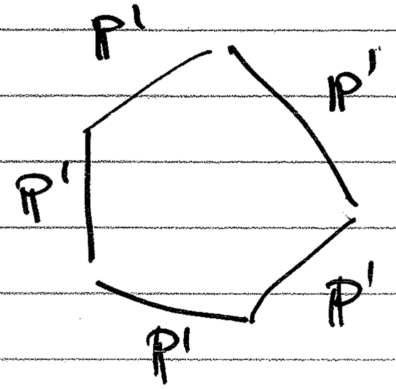


In each sector symplectomorphism

of $\mathbb{Z}[[x_1, \dots, x_n]][x_i^{-1}]$

which preserve $\sum \omega_{ij} \, d\log x_i \wedge d\log x_j$

then roughly speaking, glue formal scheme



for each domain, have transformation

imagine that $\mathbb{P}^1$ in toric variety.

~~apply~~ glue, & apply symplectomorphism along edges.

moduli of something on this is same as

the ~~gen~~ global functions on our affine scheme

Not alg. variety, but from our variety, remove something of codim 2

Examples: basic example

1) non-algebraic; focus-focus singularity
fibers are polarized abelian varieties.

Base $B = \{q \in \mathbb{C} \mid |q| < 1\}$

fiber at $q \neq 0$ $\quad \pi^{-1}(q) = \mathbb{C}^\times / q^{\mathbb{Z}}$ Take ell. curve

$q = 0$, get $\mathbb{CP}^1/_{0=\infty}$.

sympl. form $\quad \omega^{2,0} = \frac{dx}{x} f(q) dq \quad , f \in \mathcal{O}(\substack{\text{unit} \\ \text{disc}})^\times$

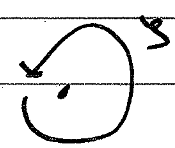
$x \in \mathbb{C}^\times / q^{\mathbb{Z}}$

all depends on $f \in \mathcal{O}(\mathbb{D})^\times$, so inf. family easily see its S-W.

$$X^\vee_\zeta := \operatorname{Spec} \mathbb{Z}[x,y,z]/_{(xy-1)z=1}$$, complement to hyperbola in plane

$$\omega_{X_\zeta} = \frac{dx\,dy}{xy-1}$$

if rotate ζ, get automorphism

monodromy 

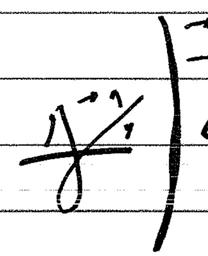
is $x,y,z \longmapsto xz, \frac{y}{z}, z$.

only non-alg. example known.

2) (Generalized) Hitchin system:

suppose Y smooth alg. $/\mathbb{C}$, and

in $T^*Y \supset$ closed reduced L
alg. Lagr. subvariety

 $L = L_0$

$$\text{"}\left(\frac{d}{dt}L_t\right)\text{"} \in \Omega^1(L^{smooth})$$ closed.

L_t : move L_0.

$\cup$

finite-dim'l subspace: some compactification

$$\Omega^1(\bar{L}^{smooth})$$

L not completely fill, should add something's pt

(doesn't depend on ~~subspace~~ compactification b/c global hol. forms is birational invariant).

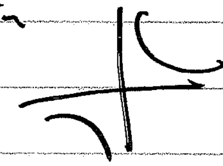
Claim: $\exists$ canonical family of L_u) $u \in B$ smooth alg

such that $T_uB \simeq \Omega^1(\bar{L}_u^{smooth})$.

Then one can make a bundle of this family:

$$\begin{array}{ccc} X & & \mathrm{Pic}\left(\bar{L}_u^{\text{smooth}}\right)^\vee \\ \downarrow & & \downarrow \\ B & & u \end{array}$$

T fiber dual T Base, so naturally carries symp. form



why of SW type?

Pic. lattice, $H_1(\bar{L}_u^*, \mathbb{Z})$)

SW gives $Z_u : \Gamma_u \longrightarrow \mathbb{C}$ $\qquad \alpha = p\,dq \in \Omega^1(T^*Y)$

$u \in B$. $\qquad d\alpha = \omega_{T^*Y}$

$$\left[\alpha\big|_{L_u^{\text{smooth}}}\right] \in H^1\left(L_u^{\text{smooth}}\right)$$

$$\cup\!/$$

$$\frac{\partial}{\partial u} \qquad \in \Omega^1\left(\bar{L}_u^{\text{smooth}}\right)$$

choose some basepoint, get Z.

e.g. Y curve,

L spectral curve,

moduli of spectral curves base B,

fiber = moduli of line bundles of spectral curve.

Hitchin system in standard sense.

Different Y, L give same i.s.

dim Y plays no role?!

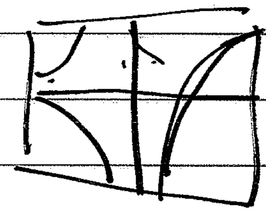
Invent crazy new Lag'n in $\mathbb{R}^n \rightsquigarrow$ new Hitchin system

Y = curve C

$B = \{$ spectral curves $L \subset T^*C \}$ — curve could be cpct, or ramified case

T^*C

e.g.



could go to ∞

$\exists \ (T^*C)' \supset T^*C$

alg. sympl.

$\bar{L}$ extends to ~~cpct~~ cpct. smooth curve in $(T^*C)'$.

$X^\vee_\zeta$ = moduli ~~of~~ space of local systems on C (unramified, when $L \subset T^*C$ really cpct)

if goes to ∞, fix some Stokes data

$\Rightarrow$ get something finite

or on C - punctures + stokes data, e.g. (pts. where L goes to ∞).

+Stokes data: $e^{\frac{\sum c_{i,\alpha}(z-z_i)}{\zeta}}$

degree of L on base is rk of local syst.

unclear geometry of walls: l.c. finite or rel. dense??

Idea of (G—)?: SL_2 loc. systems w/ singularities:

big open spaces w/ no walls

then the formal series are algebraic coordinates

If walls are dense, the complement is uncountable, so (e.g. series is divergent)

don't get alg. geom.

our variety is a domain in a torus, so sympl. form has periods

$\omega^{2,0}$. $\int_{\text{cycles } X^\vee_\zeta} \omega^{2,0} \in (2\pi i)^2 \mathbb{Z}$, so non-zero.

which indicates:
should be upgraded to an element of $K_2^{\text{milnor}}(X^\vee_\zeta)$

F field of rat'l fns

$K^2(X^\vee_\zeta)$ embeds in $K_2(F) = \Lambda^2(F^\times) / a \wedge (1-a)$ ← Steinberg rel'n.

this maps to 2-forms on Zariski open part, says should extend

then periods automatically in $(2\pi i)^2$

much more than two-form, e.g.
$\forall$ root of 1, $\zeta^N = 1$, should get
a bdle of Azumaya algebras, of rk $= N^2$ & Br
class in Brauer gp. comes from class in K^2.

true in any dimension.
Conjecture: K^2 embeds in forms, & form should have no poles

For Hitchin systems, $\exists$ different alg. structure on $X^\vee_\zeta(\mathbb{C})$

$\left\{ \begin{array}{c} \text{alg. vector bundles on } \mathbb{C} \\ \text{w/ connections} \end{array} \right\}$ then no longer affine variety.

then one can make semi-twistor family completely algebraic $M_{DR}(\mathbb{C}) \simeq M_B(\mathbb{C})$

$X_\zeta^\vee(\mathbb{C})$.

$\downarrow M_{DR,\zeta}$ as $\zeta \to 0$ get Higgs bdles

$\xrightarrow[\mathbb{C}]{} \zeta = \hbar$ $\quad B \downarrow$ dual to integrable system

$\downarrow$ P.i

(spectral curves)

class: one also gets flat coordinates here.

Namely, consider some section of this guy.

$$\mathcal{E}_\hbar,\ \nabla_\hbar = \frac{d}{dx} + \hbar^{-1} \underset{\text{Higgs field.}}{A_{-1}} + A_0 + \text{—} \quad \text{first order pole}$$

$/ \mathbb{C}[[\hbar]]$

its a module over deformation-quant. of cotangent bdle:

$$\left(\hbar \frac{d}{dx},\ x\right)\ ((\hbar)).$$

on cotangent bdle, get module over sheaf of algebras, which is supported on a spectral curve:

get sheaf of $A / \mathbb{C}[[\hbar]]$ on $T^* C$

sheaf of modules supp'd on spectral curve L,

$$\underset{\text{in analytic topology}}{\sim}\ \mathcal{O}(X)\,[[\hbar]]$$

(b/c micro-differential operators exist —)

unique sheaf of rk. 1.

two such sections differ by act, $\in H^1_{\text{analytic}}(L, \mathbb{C}[[\hbar]]^\times)$

now decorate.

so difference between 2 pairs $\Rightarrow$ elts. is some torus.

independent as choice of directions

(just requires char $= 0$, completely alg.)

Convergence? big doubts, no idea.

& how is it related to flat coords away from geom,

wall crossing.