Neitzke I:

Joint w/ D. Gaiotto, G. Moore

- A new scheme for constructing HK metrics on the total spaces of $\mathbb{C}$-int. systems (w/ extra data $\Omega(\gamma)$ that obey a wall-crossing formula)

- By twistor theory (Hitchin, HKLR)

HK mfold means

" $\mathbb{C}$ sympl. mfold over $\mathbb{CP}^1$ " (of a special sort).

ζ

Restrict to $\zeta \in \mathbb{C}^*$,

our constructions will look like gluing together patches which are open subsets of $(\mathbb{C}^*)^{2r}$

[cf. Kontsevich-Soibelman, Gross-Siebert, Auroux, Gross-Hacking-Keel, —]

Novelty: keep track of ζ-dependence.

[fibers over 0, ∞ very different, not gotten from]

particularly the asymptotics as $\zeta \to 0, \infty$

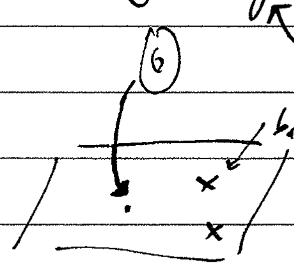
Our HK metric will take the form

$$g = g^{sf} + (\text{instanton cor.})$$

simple, explicit.

weighed by Π's of $\Omega(\gamma)$,

fix bad behavior of g^{sf} near sing. fibers.



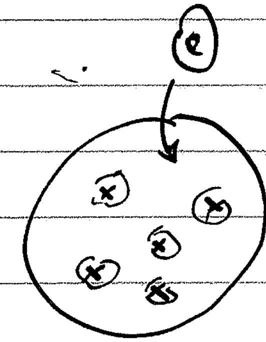
bad fibers, cod. 2, g^{sf} doesn't extend over bad fibers.

Need instanton corrections for that

Gross-Wilson:

elliptically fibered K3

generic $\rightarrow$ 24 I_1 fibers



$g = g^{sf}(R)$

$R \longrightarrow \infty$: torus fibers <u>shrink</u>.

Near each bad fiber, G-W glue-in "Ooguri-Vafa metric". corresponds to a <u>single</u> instanton correction. exponentially close as $R \to \infty$

Plan: 1) Twistor geometry of $\mathbb{R}^2 \times T^2$

Ooguri-Vafa

2) general instanton corrections (KS, WCF)

3) Hitchin systems

<u>HK geometry</u>:

$(\mathcal{M}, g)$ kähler w.r.t. 3 sympl. str. ω_i.

$\mathbb{C}$ str. J_i.

In fact, an S^2 worth of J's: any $\sum a_i J_i$, $\sum a_i^2 = 1$.

Better, think of this as $\mathbb{CP}^1 \ni \zeta$

$\forall \zeta$, $\mathcal{M}^{(\zeta)} = (\mathcal{M}, J^{(\zeta)})$ is <u>hol. sympl.</u>

$$\omega := \frac{-i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\omega_-$$

$$[\omega_\pm = \omega_1 \pm i\omega_2]$$

enough to construct HK metric.

Take $\mathcal{M} = \mathbb{R}^2 \times T^2$, w/ flat product metric determined by $\tau \in \mathbb{H}/SL(2,\mathbb{Z})$, $\mathrm{vol}(T^2) = 1/R$

Fix a $\mathbb{C}$-str. on $\mathbb{R}^2$: $\mathbb{C}$ coordinate a.

The HK str. we're after:

$$\mathcal{M}^{(\zeta=0)} \simeq \mathbb{C} \times T^2_\tau$$

$$\mathcal{M}^{(\zeta\in\mathbb{C}^\times)} \simeq \mathbb{C}^\times \times \mathbb{C}^\times$$

Fix angular coords θ_e, θ_m ~~$\in \mathbb{R}/2\pi\mathbb{Z}$~~ $: T^2 \to \mathbb{R}/2\pi\mathbb{Z}$.

$$y = \theta_m - \tau\theta_e \qquad \left.\begin{array}{l} y \sim y + 2\pi \\ y \sim y + 2\pi\tau \end{array}\right\} \text{periodicity}$$

then define

$$\mathcal{X}_e = \exp[\pi R\zeta^{-1}a + i\theta_e + \pi R\zeta\bar{a}]$$

(complexify in a way dep. on ζ)

$$\mathcal{X}_m = \exp[\pi R\zeta^{-1}\tau a + i\theta_m + \pi R\zeta\bar{\tau}\bar{a}]$$

$$(\mathcal{X}_e, \mathcal{X}_m)(\zeta) : \mathcal{M} \to \mathbb{C}^\times \times \mathbb{C}^\times$$

$$[\zeta \in \mathbb{C}^\times]$$

[Pull back $\mathbb{C}$-str. $J^{(\zeta)}$ from $\mathbb{C}^\times \times \mathbb{C}^\times$ to $\mathcal{M}^{(\zeta)}$].
& hol. sympl. str. $d\log \mathcal{X}_e \wedge d\log \mathcal{X}_m$

$$\omega = \frac{1}{4\pi^2 R}\, d\log \mathcal{X}_e \wedge d\log \mathcal{X}_m$$

$$= \frac{-i}{2\zeta}\omega_+ + \omega_\zeta - \frac{i}{2}\zeta\omega_-$$

$$\omega_+ = \frac{1}{2\pi} da \wedge dy \qquad \text{[hol. sympl. str. on } \mathcal{M}^{(\zeta=0)}]$$

(need to rescale as $\zeta \to 0$).

$$\text{i.e. } \mathcal{M}^{(\zeta=0)} \simeq \mathbb{C} \times T^2_\tau.$$

$$\omega_3 = \frac{i}{4\pi}\left[2\pi R(\mathrm{Im}\,\tau)\, da \wedge d\bar{a} + \frac{1}{2\pi R(\mathrm{Im}\,\tau)} dy \wedge d\bar{y}\right]$$

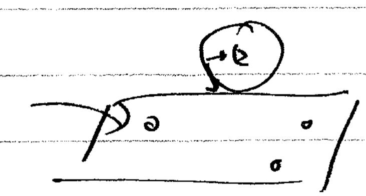
$$\cancel{\mathcal{X}_e = \exp[\pi R\zeta^{-1}a + i\theta_e + \pi R\zeta\bar{a}]}$$

now $\mathcal{X}_e$, $\mathcal{X}_m$ have essential singularity as $\zeta \to 0$.

idea: produce more interesting fns. w/ this essential singularity as $\zeta \to 0, \infty$.

when have sol'n of diff'l equation that $\mathcal{X}_e$, $\mathcal{X}_m$ satisfy, have Stokes phenomenon.

Alek Lec p73



Semiflat geometries:

Replace $\mathbb{C}$ by a (1-dim'l) $\mathbb{C}$-manifold $\mathcal{B}'$

Replace $(a, \tau a)$ in $\mathcal{X}_{e,m}$ by general hol. fxns. Z_e, Z_m: $\mathcal{B}' \to \mathbb{C}$

Now $\mathcal{X}_e = \exp[\pi R Z_e/\zeta + i\theta_e + \pi R \zeta \bar{Z}_e]$

Same construction $\rightsquigarrow$ simple HK metric

At $\mathcal{M}^{(\zeta=0)}$, torus fibers have $\tau = \frac{dZ_m}{dZ_e}$

for $\zeta \neq 0$, locally identified w/ $\mathbb{C}^\times \times \mathbb{C}^\times$.

(need assumption $\mathrm{Im}\,\tau > 0$)

Ex: Take $\mathcal{B}' =$ ~~punctured~~ $\{0 < |u| < 1\}$

$$Z_e(u) = u$$
$$Z_m(u) = \frac{1}{2\pi i}(u \log u - u).$$

(pick a cut, single-valued)

Same constr. as before:

$\rightsquigarrow$ HK metric g^{sf} on T^2-bdle over $\mathcal{B}'$.

but no way of extending over $u=0$

To extend HK str. over $u=0$:

Call our previous $(\mathcal{X}_e, \mathcal{X}_m)$ $\quad \mathcal{X}_e^{sf}, \mathcal{X}_m^{sf}$.

Improved version:

$$\mathcal{X}_e = \mathcal{X}_e^{sf}$$

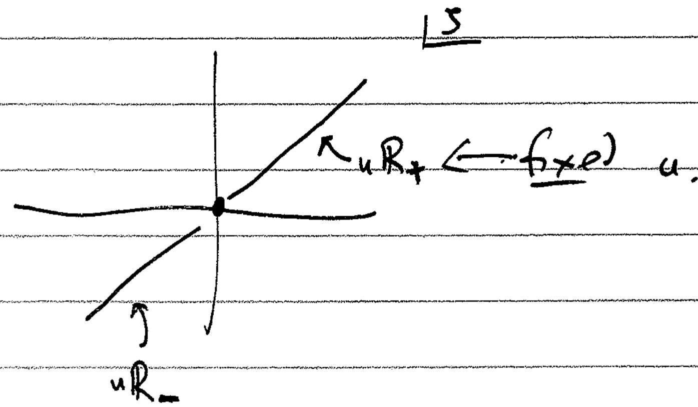
$$\mathcal{X}_m^{(\zeta)} = \mathcal{X}_m^{sf(\zeta)} \exp\left[\frac{i}{4\pi}\int_{Z_e\mathbb{R}_-} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log[1-\mathcal{X}_e(\zeta')] - \frac{i}{4\pi}\int_{Z_e\mathbb{R}_+}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log[1-\mathcal{X}_e^{-1}(\zeta')]\right]$$

$u\mathbb{R}_- \to Z_e\mathbb{R}_-$

$u\mathbb{R}_+ \to Z_e\mathbb{R}_+$

$\mathcal{X}_m(\zeta)$

$\mathcal{X}_m(\zeta)$ is discontinuous as fxn. of (u, ζ):



As ζ crosses $u\mathbb{R}_-$, $\mathcal{X}_m \to \mathcal{X}_m(1-\mathcal{X}_e)$

" " $u\mathbb{R}_+$, $\mathcal{X}_m \to \mathcal{X}_m(1-\mathcal{X}_e^{-1})^{-1}$.

[comes from $\Omega(\gamma_e)=1$, $\Omega(\gamma_m)=0$].

Despite this, $\omega = d\log \mathcal{X}_e \wedge d\log \mathcal{X}_m$

is perfectly smooth, and

still has $\omega = \frac{1}{\zeta}\omega_+ + i\omega_3 + \zeta\omega_-$

The correction to $\mathcal{X}_m$ is finite in the limit $\zeta \to 0$.

("good asymptotics").

This $\mathcal{X}_m$ $\zeta = 0$ simple asymptotic form $\sim \exp(\pi R Z_u/\zeta)c$,

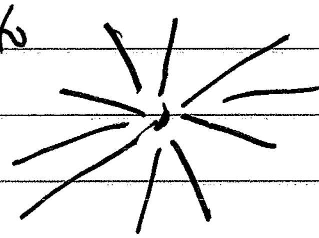
exhibits Stokes phenomena.

"analytic continuation of asymptotics $\neq$ asymptotics of analytic continuation"

$\mathcal{X}_m$ has Stokes phenomena!

Can calculate metric explicitly: Ooguri-Vafa metric:
only ingredient is the symplectomorphism.

Idea: generalize to



pile of Stokes factors

when these rays hit each other, get W-C-F.