T. Pantev. MS and Mixed Hodge structures

joint w/ Katzarkov & Kontsevich

Recall: A pure nc HS on V is the data

$$((\mathcal{H},\nabla), \mathcal{S}_{(\mathbb{Q})}, \underline{\mathrm{iso}})$$

$\mathcal{H} \to \mathbb{A}^1$: holo. v.b. s.t. $\mathcal{H}_1 \cong V$.

$\mathbb{Z}_2$-graded v.b.

$\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1$ - meromorphic with ≤ 2 order pole

$\mathcal{S}_{(\mathbb{Q})}$ - local sys. of $\mathbb{Q}$-v.s. on $\mathbb{A}^1 - \{0\}$

$\underline{\mathrm{iso}}$: $\mathcal{S}_{\mathbb{Q}} \otimes \mathbb{C} \xrightarrow{\sim} \mathcal{H}_1|_{\mathbb{A}^1-\{0\}}$

satisfying a tricky list of axioms

Sources: If C is a smooth, cpct dg category over $\mathbb{C}$, then $HP_\bullet(C)$ supports a natural pure nc HS.

↑ defined over $k((u))$ ← this is the $\mathbb{A}^1$ parameter

Example: (X,ω) - cpct. sympl. $\dim_{\mathbb{R}} X = 2d$,

There "is" a nc. H-S. on $H^\bullet(X,\mathbb{C})$

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1; \quad \mathcal{H}^0 = \left(\bigoplus_{\substack{k=d \\ \mathrm{mod}\ 2}} H^k\right) \otimes \mathbb{C}\{u, q\}$$

variation of Hodge structure (from large vol. limit)

$$\mathcal{H}^1 = \left(\bigoplus_{\substack{k=d+1 \\ \mathrm{mod}\ 2}} H^k\right) \otimes \mathbb{C}\{u, q\}$$

$$\nabla_{\frac{\partial}{\partial u}} = \frac{\partial}{\partial u} - u^{-2} K_X \underset{q}{\overset{\text{quantum prod}}{*}} (\cdot) - u^{-1} Gr_X, \qquad Gr_X|_{H^k} = \frac{k-d}{2}\,\mathrm{id}$$

$$\nabla_{\frac{\partial}{\partial q}} = \frac{\partial}{\partial q} - u^{-1} q^{-1} [\omega] *_q (\cdot)$$

$\mathcal{E}_{\mathbb{Q}}$: image

$$H^{\cdot}(X;\mathbb{Q}) \xrightarrow{(2\pi i)^{\cdot/2}} H^{\cdot}(X,\mathbb{C}) \xrightarrow{\hat{\Gamma}(X) \wedge \cdot} H^{\cdot}(X,\mathbb{C})$$

$$\hat{\Gamma}(X) = \prod_{i=1}^{q} \Gamma(1+\delta_i)$$

characteristic class

chern roots of (T_X, J)

Problem: this connection not clearly defined over $\mathbb{A}^1$.

☆ Fukaya category being saturated & this thing converging over $\mathbb{A}^1$.

strange connection ???

Known existence restricts GW theory ??

For toric Fanos, all this has been checked

Mixed theory: what sort of symplectic data should one get?

Q: what are nc-mixed HS, &

what sympl. top. gives rise to MHS ??

Recall: where do MHS come from in Kähler geometry?

cohomology of X: X-non-cpct

X-non-smooth,

asymptotics $\begin{array}{c}\mathcal{X}\\ \downarrow\\ \Delta\end{array}$ family of compact kähler mfolds degen. at $0 \in \Delta$

$\mathcal{X}_t$ smooth $\to H^\cdot(\mathcal{X}_t)$ has a pure HS

$H^\cdot(\mathcal{X}_0)$ has a MHS, & also we have a continuous

map $r : \mathcal{X}_t \to \mathcal{X}_0$. comes from retracting

~~Gauss~~ non holomorphic but asymptotically is & so induces a map of H.S.

from $H^\cdot(\mathcal{X}_0) \to H^\cdot(\mathcal{X}_t)$

w/ modified H.S. here.

<u>Assume</u>: $\begin{array}{c}\mathcal{X}\\ \downarrow\\ \Delta\end{array}$ - standard degen. ($\mathcal{X}_0$ - strict norm. cross, T = unipotent).

$\Rightarrow \log(T) = N : H^\cdot(\mathcal{X}_t) \circlearrowleft$ induces a weight filtration W (monodromy weight filtration).

Clemens - Schmid :

$$\left(H^\cdot(\mathcal{X}_t), W, \lim_{q \to 0} e^{N_2} F \right)$$

a mixed hodge structure

Want a sympl. analogue of this construction.

<u>Use mirror symmetry!</u>

Suppose we have

$$\begin{array}{ll} \mathcal{X} & \text{complex CY of dim. } n. \\ \downarrow f & \text{proper} \\ \Delta & \end{array}$$

f has a simple singularity

$\mathcal{X}_0$ strict normal crossings, unipotent monodromy

Want to understand the mirror of this:

Q: - what is the sympl. geometry of a mirror of this data?
- what is the mirror of the limit MHS.

One case we understand: maximally unipotent degen.
By studying examples the mirror is:

B-model

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \Delta \end{array} \quad \xleftrightarrow{\text{MS}}$$

$(\dim_{\mathbb{C}} \text{total space} = n)$

A-model

"family" of sympl. mflds associated to (Y, ω, D)

Y cpt. of dim'n $2n-2$
ω - sympl. form
D smooth divisor in Y
really a family of Fukaya categories.

We have a family of Fukaya categories parametrized by Δ

$$\mathcal{F} \longrightarrow \Delta$$

$$\mathcal{F}_q = \mathrm{Fuk}(Y, [\omega] - \log q \, PD[D])$$

$$\mathcal{F}_0 = \mathrm{Fuk}^{\text{wrapped}}(Y - D, \omega)$$

"wall of kähler cone" defined by D.

(Y c-Y. D some divisor in Y.)

<u>HMS:</u>

<u>B-side</u>	<u>A-side</u>
$D^b(\mathcal{X}_q)$	$\mathcal{F}_q$.
$D^b(X_0)$	$\mathcal{F}_0$
$\mathrm{Perf}(\mathcal{X}_0)$	$(\mathrm{Fuk}(Y-D, \omega))$
$D^b_{sing}(\mathcal{X}, f)$	$\mathrm{Fuk}(D, \omega_D)$

<u>Also:</u> Monodromy on $H^\cdot(\mathcal{X}_t)$ is mirrored by

$$N = PD[D] *_q.$$

limit MHS is

$$(H^\cdot(Y), W \text{ for } N, \lim e^{N_\sharp} \nabla)$$

actually $HP^\cdot$

In the opposite direction:

$$
\begin{array}{ll}
\underline{\text{A-model}} & \underline{\text{B-model}} \\
\mathfrak{X} & \\
\downarrow f & (Y, D) \\
\Delta & \\
\mathrm{Fuk}(\mathfrak{X}_t, \omega) & D^b(Y) \\
\mathrm{Fuk}(X_0, \omega) & D^b(Y - D) \\
FS(X, f, \omega) & D^b(D)
\end{array}
$$

(" CY w/ variety of general type embedded in it ")

not really defined currently.

$\mathrm{Fuk}(X_0, \omega)$ is obtained from $\mathrm{Fuk}(X_t, \omega)$ by quotienting by $T : \mathrm{Fuk} \circlearrowleft$ (T - spherical?) limits D's.

and need $\varphi : \mathrm{Id} \longrightarrow T$.

Can be worked out when Φ - dual intersection complex of central fiber $\mathfrak{X}_0$, is a graph

(in higher dimensions, have to use LG mirrors w/ many potentials)

e.g. $\mathfrak{X}$ semistable degeneration of elliptic curves
$\mathfrak{X}$ - type IV degeneration of K3.

Start w/ Fano w/ anticanonical D smooth, poss. disconnected
mirror : LG w/ disconnected fibers.
each component is vertex, look at star of vertex. boundary of star of vertex

defn smoothing to first order as a log structure or on
B.O. of normal bundle of one component w/
conormal of other.

Quantum product w/ [D] gives spherical functor which
are inverse to monodromy
b/c isomorphic, can glue.
⇝ get LG over $\mathbb{P}^1$ or Ell. curve.,
broken into pieces according to dual intersection complex.

Can get it in opposite direction (Abouzaid-Auroux-Katzarkov)

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applications: filtrations on SH^*??

Mixed HS, only [D] enters.

<u>Mohammed:</u> Idea is that $HP^*(\ldots)$ or that version of SH
should be deformation invariant; if so would imply that
Fuk is never 0 when vary sympl. form

T^*S^2: when turn on $t[S^2]$, everything vanishes??
Maybe curved lagrangians survive.