Schapira II:

M real m'fold $F \in D^b(k_M)$

$SS(F) \subset T^*M$

- closed, $\mathbb{R}^+$-conic
- $SS(F) \cap T^*_M M = \operatorname{supp} F$
- & co-isotropic

Z closed conic subset of T^*M

$D^b_Z(k_M) \subset D^b(k_M)$ $\quad U = T^*M \setminus Z$.

$= \{F;\ SS(F) \subset Z\}$

$$D^b(k_M, U) = D^b(k_M) / D^b_Z(k_M)$$

$$Ob(D^b(k_M; U)) = Ob(D^b(k_M))$$

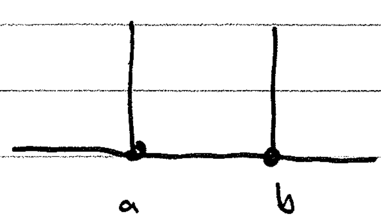
$u : F_1 \to F_2$ is an iso. on U.

iff $\quad ; F_1 \xrightarrow[u]{} F_2 \to F_3$

& $SS(F_3) \subset Z$

Ex: $M = \mathbb{R}$, T^*M (t, τ)

$U : \tau > 0$.



$F_a = k_{\{a\}} \simeq k_{[a, +\infty)}$

work in $D^b(k_{\mathbb{R}}; \tau > 0)$

$F_b = k_{\{b\}} = k_{[b, +\infty)}$

$F_a \xrightarrow[u]{} F_b$

$$u \neq 0 \text{ in } D^b(k_{\mathbb{R}}, \tau > 0)$$

examples of two lag'n that don't intersect, but there <u>are</u> morphisms.

when you localize, can pick up non-local phenomena.

classic trick to make conic:

$$T^*M \times T^*_{\tau>0}\mathbb{R} \xrightarrow[\rho]{} T^*M$$

$$(x, \xi, t, \tau) \longmapsto (x; \xi/\tau)$$

$$D^b(k_M^\tau) := D^b(k_{M\times\mathbb{R}}; \tau > 0)$$

$A \subset T^*M$

$$D^b_A(k_M^\tau) := D^b_{\rho^{-1}(A)}(k_{M\times\mathbb{R}}; \tau > 0)$$

Tamarkin's idea: work in this category (not exactly, need to kill torsion)

<u>Lemma</u>:

$$D^b(k_{M\times\mathbb{R}}; \tau > 0) \simeq \left(D^b_{\{\tau\leq 0\}}(k_{M\times\mathbb{R}})\right)^{\ell\perp}$$ ← left orthogonal

$$\cong$$

$$\{F \in D^b(k_{M\times\mathbb{R}}),\ F \simeq F \star k_{\{t\geq 0\}}\}$$

$$(M\times\mathbb{R})\times(M\times\mathbb{R}) \longrightarrow M\times\mathbb{R}.$$

~~convolution~~ Have natural morphisms

$$F \longrightarrow T_c F = F * k_{[c,+\infty)}$$

$$k_{[0,+\infty)} \to k_{[c,\infty)}.$$

<u>Torsion object</u> $F \in D^b(k_M^\sigma)$

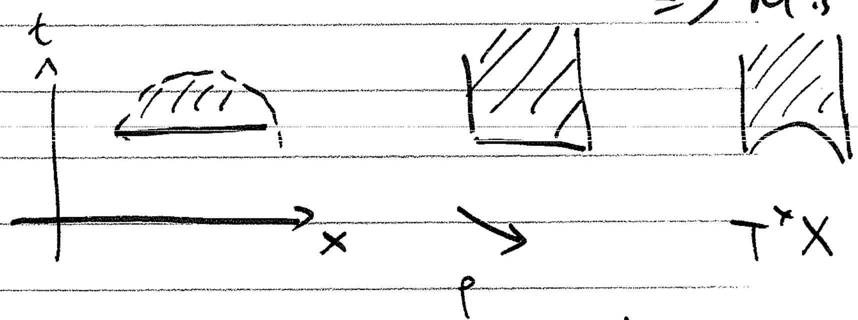
F is torsion if $\exists\, c \geq 0$

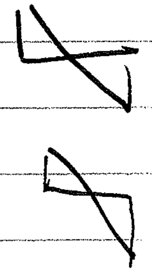
(*) $F \longrightarrow T_c F$ in $D^b(k_{M\times\mathbb{R}})$

$$F \searrow 0 \nearrow T_c F$$

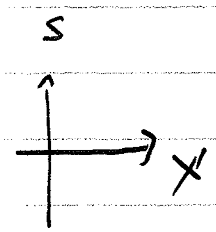
<u>Ex</u>: supp $M \to F$ is proper

$\Rightarrow M$ is torsion.





$\rho(SS(k_A))$



$T^*_{\tau>0}(M\times\mathbb{R})$.

Thm 1: A, B cpct in T^*M

$A \cap B = \emptyset$

$$F \in D^b_A(k_M^\tau) = D^b_{\rho^{-1}(A)}(k_{M\times\mathbb{R}}, \tau>0)$$

$$G \in D^b_B(-) \Rightarrow \mathrm{Hom}_{D^b(k_M^\tau)}(F, G) = 0.$$

difficult b/c many operations like convolutions are non-proper.

(torsion Seidel: phenomenon occurs in Fuk as well, if don't localize deformation parameter q)

Thm 2: Invariance by Hamilt. isotopy

$$\phi = \{\varphi_t\}_{t\in I} \quad I \ni 0 \quad \varphi_0 = id$$
$$I \supset [0,1]$$

Torsion object $F \in D^b(k_M^\tau)$

F is torsion if $\exists\, c \geq 0$ s.t. $(*)$

$$\tau(A) = D^b_A(k_M^\tau)/\{\text{torsion}\}$$

$$\tau(M) = D^b(k_M^\tau)/\{\text{torsion}\}$$

~~$\tilde\varphi$ eq. $\tau(A)$ and $\tau(\varphi_1(A))$~~ $\tilde\varphi_1(F) \simeq F$ in $\tau(M)$

$$\Rightarrow \mathrm{Hom}_{\tau(M)}(G, F) \simeq \mathrm{Hom}_{\tau(M)}(G, \tilde\varphi_1^{-1}(F)).$$

Cor: Non-displaceability:

Assume $\exists$

$$F \in D^b_A(k_M^\tau),\ G \in D^b_B(k_M^\tau),$$

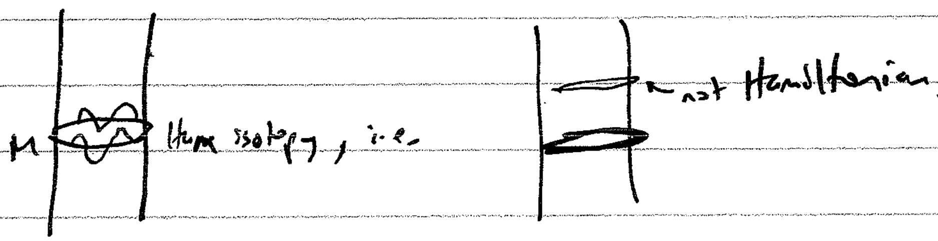
$\mathrm{Hom}_{\tau(M)}(G, F) \neq 0$, then (A, B) is non-displaceable.

<u>Lemma:</u>

$$D^b(k_M) \xrightarrow{\text{fully faithful}} \mathcal{T}(M)$$
$$F \longmapsto F \boxtimes k_{\{t \geq 0\}}.$$

Assume M compact $\qquad F = G = k_M.$

$\Rightarrow T^*_M M$ is non-displaceable



"Fukaya (T^*M)" $\simeq \mathcal{T}(M) = D^b(k_{M\times\mathbb{R}}; \tau > 0)/\text{torsion}$

Joint w/ Guillermou - Kashiwara - S on arXiv

HHI homogeneous hamiltonian isotopy

$$\Phi : \dot{T}^*M \times I \longrightarrow \dot{T}^*M$$

$\phi = \{\varphi_t\}_{t\in I} \quad \varphi_0 = \mathrm{id}$

$\varphi_t = \mathbb{R}^+$-hom. symplectomorphism

$\alpha_M =$ Liouville

$f = \langle \alpha_M, \frac{\partial \phi}{\partial t} \rangle, \quad \frac{\partial \Phi}{\partial t} = H_f$

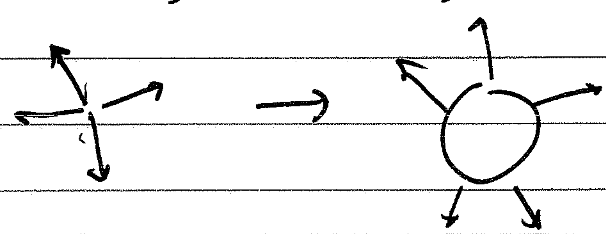
Graph $\Gamma_\phi \subset \dot{T}^* M \times \dot{T}^* M \times I$

$\exists \Lambda \subset \dot{T}^* M \times \dot{T}^* M \times T^* I$.

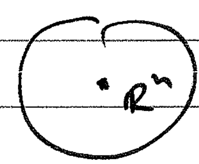
Ex: $T^* \mathbb{R}^n$ (x, ξ)

$\varphi_t : (x; \xi) \longmapsto (x + \frac{t\xi}{\sqrt{\xi^2}}; \xi)$

$\dot{T}^*_{\{0\}} M \longrightarrow \dot{T}^*_{\bar{B}(0,t)} M$



$\mathbb{R}^n \subset \mathbb{C}^n$



$\Phi = \{\varphi_t\}_{t \in I}$ $0 \in I$

$\Lambda \subset \dot{T}^* M \times \dot{T}^* M \times T^* I$

Thm: $\exists K \in D^b(k_{M \times M \times I})$

(1) $SS(K) \subset \Lambda \cup \{\text{zero-section}\}$

(2) $K|_{t=0} = k_\Delta$

(3) $\operatorname{supp} K \rightrightarrows M \times I$ proper — comp. of kernels

(4) K^{-1} $K^{-1} \circ K = K \circ K^{-1} = k_{\Delta \times I}$

(5) K satisfying (1) and (2) ~~is unique~~. has unicity

prove the resulting nbhd of $t=0$.

ND theorem: $d\psi(x) \neq 0 \ \forall x$

$\psi : M \to \mathbb{R} \quad \Lambda_\psi = \{(x; d\psi(x))\}$

ϕ a HHF $\{\varphi_t\}_{t \in I}$ $\varphi_0 = \mathrm{id}$

$F_0 \in D^b(k_M)$ ① cpct supp. ② $R\Gamma(M; F_0) \neq 0$

~~② $R\Gamma(M; F_0) \neq 0$ $\Rightarrow \varphi_t(SS(F_0)) \cap \Lambda_\psi \neq \emptyset$~~

$\Rightarrow \varphi_t(SS(F_0)) \cap \Lambda_\psi \neq \emptyset. \ \forall t.$

For arnold, take

$N \times \mathbb{R} = M, \quad \psi(x,t) = t$

Proof: K quantization of ϕ

$K_{t_0} = K|_{t=t_0} \qquad F = K \circ F_0$

$\qquad\qquad\qquad\qquad F_t = K_t \circ F_0$

$\varphi_t(\dot{SS}(F_0)) = \dot{SS}(F_t)$

$SS(F) \cap T^*_M M \times T^* I \subset$ {zero-section}

$q : M \times I \to I \quad Rq_* F \in D^b(k_I)$ const. sheaf

$\Rightarrow R\Gamma(M; F_t) \simeq R\Gamma(M; F_0) \neq 0.$

$R\Gamma(M; G) \neq 0$

G has cpct. supp. $\Rightarrow SS(G) \cap \Lambda_\psi \neq \emptyset.$

$\Lambda_\psi, \ \phi, \ \Lambda \ F_0$

$\Lambda_\psi \cap \dot{SS}(F_0) \subset S_0 \subset SS(F_0)$, S_0 is smooth, Lagrangian.

$\Lambda_\psi \cap S_0$ is transversal.

$+ F_0$ is simple in Λ_0.

$\Rightarrow \varphi_t(SS(F_0)) \cap \Lambda_\psi \geq \sum_\gamma b_\gamma(F_0)$

$\phi = \{\varphi_t\}_{t \in I}$ HHI

$\Lambda \subset \dot{T}^* M \times \dot{T}^* M \times T^* I$

Def: φ is positive if $\Lambda \subset \{\tau \leq 0\}$

$\frac{\partial \Phi}{\partial t} = H_f \quad , f \geq 0$

Thm: M connected non-cpct; X, Y two smooth connected submanifolds of M.

φ is a positive HHI

$\varphi_1(\dot{T}^*_X M) = \dot{T}^*_Y M$

$\Rightarrow X = Y$.