2/14/2013 - to scan.

Abouzaid I:

Mirror symmetry (version 0)
(without corrections).

M symplectic manifold

$\mathcal{L} :=$ space of embedded Lag'n submanifolds up to Hamiltonian isotopy.

Locally: Every $L \in M$ has a neighborhood symplectomorphic to open in T^*L.

Lagrangian sections of $T^*L \longleftrightarrow$ closed forms

Passing to Hamiltonian isotopy $\rightsquigarrow$ mod out by exact forms

$\Rightarrow \mathcal{L}$ is locally modelled after $H^1(L, \mathbb{Z}) \otimes \mathbb{R}$.

Canonical lattice is $T\mathcal{L} \rightsquigarrow$ integral affine structure.

$\mathcal{L}^{\mathbb{C}}$ complexification, pairs (L, ∇)

∇: $U(1)$ local system

Locally $\boxed{H^1(L, \mathbb{Z}) \otimes \mathbb{C}}$

Note: $\mathcal{L}^{\mathbb{C}}$ carries a canonical quadratic volume form
(b/c preserved by transition functions in $GL(n, \mathbb{Z})$)

Mirror symmetry: Find a scheme Y (analytic, non-commutative, derived) such that
"moduli of sheaves on Y" corresponds to $\mathcal{L}^{\mathbb{C}}$

Always true that $\boxed{2c_1(Y) = 0}$.

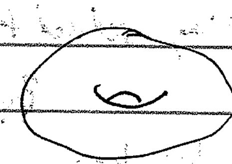
Why? Because $Y \subset \mathcal{L}^{\mathbb{C}}$.

↑ skyscraper sheaf of points, and $2c_1(\mathcal{L}^{\mathbb{C}}) = 0$.

Example: $M = T^2$ with area A.

$$\mathcal{L}_{\mathbb{C}} = \coprod_{\substack{\alpha \in H_1(T^2,\mathbb{Z}) \\ \text{primitive}}} \mathcal{L}_\alpha$$

(Lag'ns need to represent primitive classes)



(pretend no curves in 0 homology.)

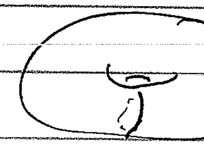
~~Think of α as a numerical invariant~~

~~Choosing a frame, & α class $= (0,1)$~~

Then $\mathcal{L}_\alpha \cong \mathbb{R}/A\cdot\mathbb{Z}$ ← this has canonical basepoint 0

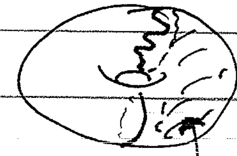
↑ area

maybe e^A.



choose basepoint.

Given any Lag'n homologous, assign # = to area between these two $\in \mathbb{R}/A\cdot\mathbb{Z}$. w/ some cylinder..



Flux.

$$\mathcal{L}_\alpha^{\mathbb{C}} = \mathbb{R}/A\mathbb{Z} \times \underbrace{\mathbb{R}/2\pi i\mathbb{Z}}_{\text{monodromy of local system on } S^1}$$

This is the mirror (up to $A \mapsto e^A$).

Other components should give you different moduli spaces of sheaves over

Note: In $M = T^2$ case,

mirror has an action by $S^1 \times S^1$ (adding real and imaginary #'s mod $\mathbb{C}$).

Obs: - The "real" action on the mirror gives rise to symplectomorphisms.

- But, the imaginary action on the mirror gives $u(1)$ local systems on M.

i.e. given ~~a~~ local system on M, send

$$(L, \eta) \longrightarrow (L, \alpha|_L \otimes \eta)$$

In general there's no reason for an action on B side to ~~a~~ correspond to intertwined actions of symp. + loc. sys. --

Infinitesimal level:

$$\mathbb{R}\text{-action on } Y \longleftrightarrow \text{Holomorphic vector field } H^0(Y, T_Y)$$

$$\longleftrightarrow \text{Deformation of } \Delta_Y \subset Y \times Y \qquad H^0(Y,T_Y) \subset \mathrm{Ext}^1(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$$

$$\mathrm{Ext}^1(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \underset{\text{HMS}}{\cong} HF^1(\Delta_M, \Delta_M)$$

Lagrangian in $M \times M^-$.

$$\mathrm{Ext}^1(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = H^0(Y, T_Y) \oplus H^1(Y, \mathcal{O})$$

If M is closed, this $\overset{\text{112}}{=}$ $H^1(M, \mathbb{C}) \cong H^1(M, \mathbb{R}) \oplus H^1(M; i\mathbb{R})$

gives category action by tensoring w/ line bundle

target of symplectomorphisms and ~~...~~ Ham. isotopy

Tangent space classifies local systems on M.

(Something needs to be checked to see the correspondence btw <u>Group actions</u> (Global case))

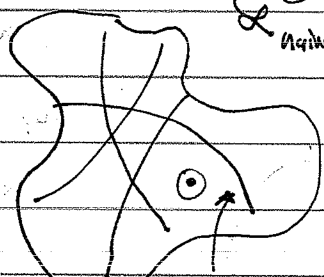
More generally: consider M open (e.g. affine variety carries a canonical sympl. form pull-back of standard form on $\mathbb{C}^N \supset M$).

Mirror symmetry with corrections:

$\mathcal{L}^{\mathbb{C}}$ define this as a space of pairs (L, b) $b \in H^1(L, \mathbb{C})$ satisfying:
a Maurer-Cartan equation, up to equivalence in the Fukaya category.

Kontsevich-Soibelman

$$\mathcal{L}^{\mathbb{C}}_{\text{naive}} = (L, \nabla)$$



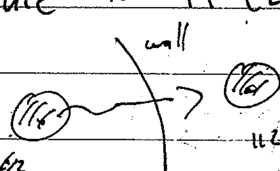
walls coming from hol. discs with boundary on L.

Reglue complements of the walls by a non-trivial map that depends on discs.

This is locally modelled after $HF^1(L, b)$ ← Floer cohomology

Often this is isomorphic to $H^1(L, \mathbb{C})$ BUT NOT CANONICALLY.

point:



$$H^1(L, \mathbb{C}) \longrightarrow H^1(L, \mathbb{C})$$

non-trivial automorphism coming from different identifications

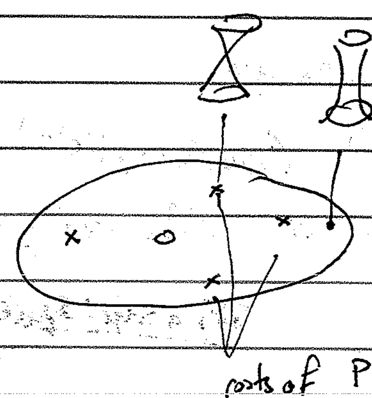
Example: (A - Auroux - Katzarkov)

$M = A_k$ Milnor fibre in dim 2 $\setminus$ Conic

$= \{ p(z) = x^2 + y^2 \mid z \neq 0 \} \subseteq \mathbb{C}^3$

$p(0) \neq 0$, $\deg(p) = k+1$, p has simple roots.

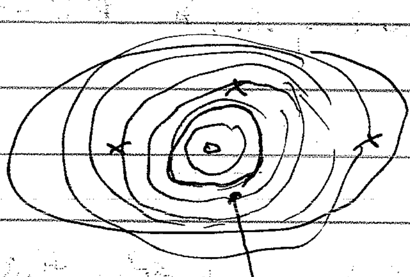
$$M \xrightarrow{z} \mathbb{C}$$



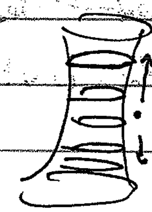
Construct a family of Lagrangian tori

$$\begin{array}{ccc} T^2 & \subset & M \\ \downarrow & & \downarrow \\ S^1 & & \mathbb{C}^* \end{array}$$

What are these circles in base?

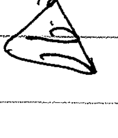


doesn't go through singularity, fiber =

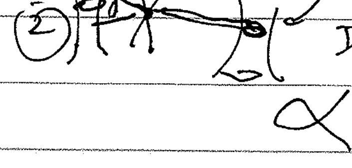


move circle of height h around puncture $\to$ 2-torus.

If $S^1 \subset \mathbb{C}^*$ includes a zero of p, then upstairs, depends on height.

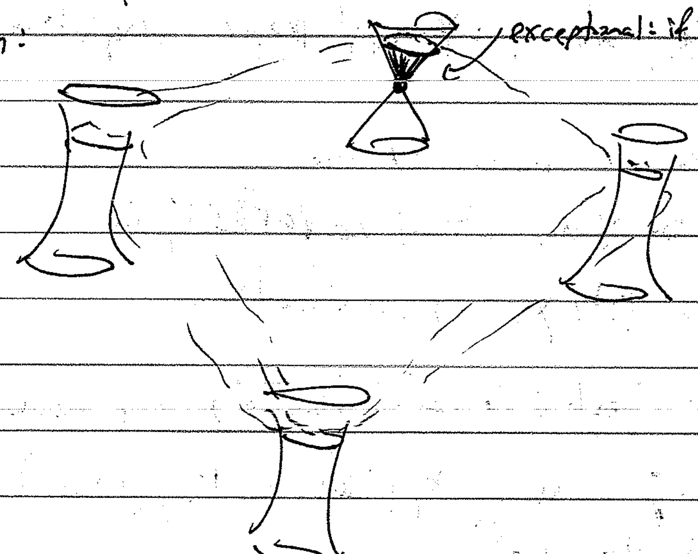


2 possibilities (1) T^2



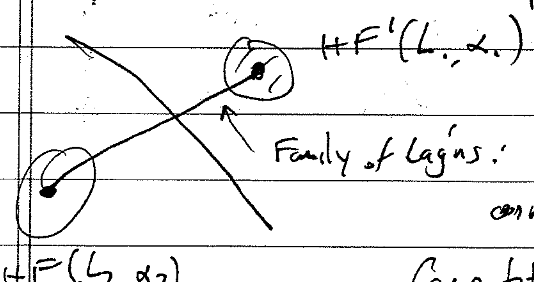
Naive idea: Consider $\mathring{\mathcal{L}}^{\mathbb{C}}$ ~~space of smooth/embedded tori $T^2 \subset M$~~
~~in this ~~fiber~~ family,~~
~~+ local systems,~~
~~and compactify at the singularities.~~
~~(Fukaya, Kontsevich-Soibelman)~~
~~This does not work.~~ ~~(No cptification)~~

Instead: consider walls in $\mathring{\mathcal{L}}^{\mathbb{C}}$ formed by tori which bound an "exceptional holomorphic disc."
These are the codim-1 locus of tori such that the projection to ~~base space~~ $\mathbb{C}$ includes a zero of p.
Reason:



condition of bounding this disc = codim 1

Cut $\mathring{\mathcal{L}}^{\mathbb{C}}$ open along these walls and glue via KS wall-crossing ~~formula.~~ (this is a computation.



Family of Lag'ns: Canonical way of following through HF along connection. Get an identification:
Computation: This is KS wall crossing formula.

$\rightsquigarrow$ New space $\boxed{\mathcal{L}^{\mathbb{C}}}$.

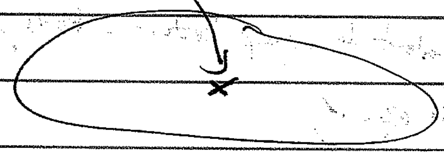
Explicit answer for ~~...~~ the mirror of $A_k \setminus$ conic $\in \mathbb{C}^* \times \mathbb{C}^2$ (Milnor over A_k)

i.e.

Let A_k (singularity) $\setminus$ Conic $\subset \mathbb{C}^* \times \mathbb{C}^2$

$$\{ z^{k+1} = x^2 + y^2 \mid z \neq 1 \}.$$

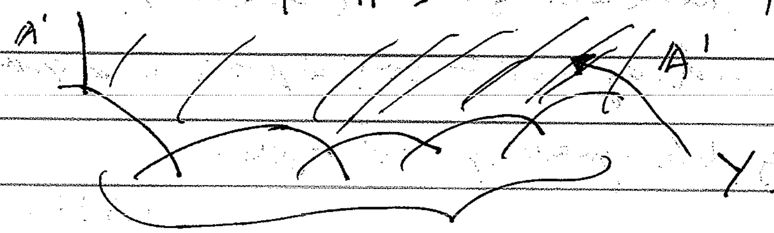
z^{k+1} very singular at 0.



Define Y to be the resolution.

Note: Y is diffeo. to M (in this example)

$\exists$ a chain of k $\mathbb{P}^1$'s embedded in Y.



Puzzle: $H^0(Y, T_Y)$ $(\oplus H^1(Y, \mathcal{O}))$ ← infinite rank

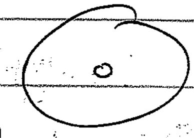
but. $\boxed{H^1(M, \mathbb{C})} \leftarrow \boxed{\text{rk } 1}$,

(eg. $k = 0 \rightsquigarrow \boxed{\mathbb{C}^2 \setminus \text{Conic}}$.)

Solution: expected isomorphic to $SH^1(M)$

↑ symplectic cohomology.

~~Y projec~~ M projects to



pull back of class here

Note: $\gamma \sim S^2$ model.

$$HF^*(\gamma; \mathbb{C}) \cong H^*(T^2, \mathbb{C}).$$

$t \longrightarrow$

(hard analysis)
Continuation: 

↑ discs at various points all [illegible]

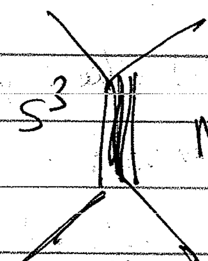
FOUO get this up to be a chain map.

Idea: push $\nu^* K$ ($\nwarrow$ not in S^3) off of S^3
using its deformations.

Gopakumar – Vafa: '98
large N duality:

$\mathcal{O}(-1) + \mathcal{O}(-1)$
$\downarrow$
$\mathbb{P}^1$

$\longleftrightarrow$



$\mathbb{P}^1$ of size $N\lambda$ ($\nwarrow$ genus)

$X^* = T^* S^3$
with $\boxed{N}$ A-Branes on S^3.

Witten: A-model on $T^* X^3$ ($\nwarrow$ 3 mfds.)
w/ N Lag'n branes on base, the
top. A-model = SU(N) C-S theory on S^3.
~~w/~~ all hol. discs to base degenerate to graphs:
SU(N) chern simons theory on S^3.

λ – genus counting parameter $\longleftrightarrow$ quantum parameter λ in CS theory
$\lambda = k \rightarrow (\lambda' = 2\pi i/k+N\ ?)$

So CS theory of S^3 $\xleftrightarrow{\text{CS duality}}$ ~~A-model~~ A-model on T^*S^3 $\xleftrightarrow{\text{large } N}$ $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ $\xleftrightarrow{\text{MS}}$ Riemann surface.
Computable if you know HOMFLY polynomial.