Abouzaid II:

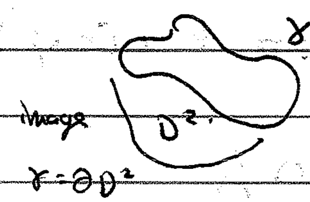
M symplectic manifold "tame at ∞"

e.g. $M = \overline{M} \setminus D$
smooth projective — divisor s.t. $D \cdot C \geq 0$.

In practice: just say M is affine.

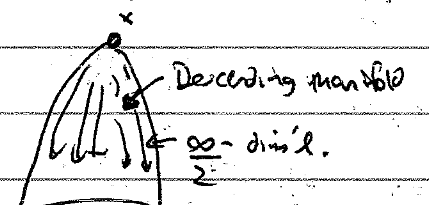
Floer: $\mathcal{L}M \xrightarrow{\text{action}} \mathbb{R}$.

$\int_{D^2} \omega$



Not necessarily well-defined, but its critical points are.
Floer homology is the study of critical points of this function.

e.g. Morse theory



Can always choose coords p, q s.t.
$\omega = \sum dp_i \wedge dq_i$
$\infty/2$ means finite in p, infinite in q.

Ex: $M = T^*Q$, Q smooth closed, & spin.

$SH^*(M) = H_{n-*}(\mathcal{L}Q)$

i.e. given γ finite dim'l family of loops in Q, consider all loops in M which project to γ.
(exactly ~~half~~ f.d. in Q, inf. dim'l in p).

If M is closed, then crit. pts. of action functional is a copy of $\underset{\text{const. loops}}{M \subset \mathcal{L}M}$, gives

$$SH^*(M) = HF^*(M) = QH^*(M).$$

Connections with Fukaya categories:

$\mathcal{F}$

$\mathcal{W}$ $\quad Ob(\mathcal{F})$ = closed Lagrangians in M + ...

$Ob(\mathcal{W})$ = Properly embedded Lagrangians in M

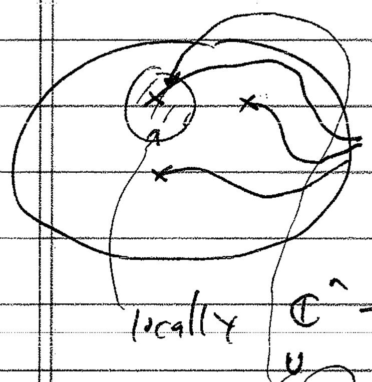
wrapped which are "tame at ∞" + technical conditions.

obviously, $\mathcal{F} \subset \mathcal{W}$

example: ① Real part of affine variety if it's smooth.

② Given a hol. function $M \longrightarrow \mathbb{C}$ with

"Lefschetz fibration" generic singularities (locally $(z_1, \dots, z_n) \mapsto \sum z_i^2$)



Lagrangians that project to arcs.

locally $\mathbb{C}^n \longrightarrow \mathbb{C}$

$\quad\quad\;\cup$

put it in $\mathbb{R}^n \longrightarrow \mathbb{R}^+$

, & extend to ∞ by parallel transport.

Called thimbles.

Affine varieties have "many" Lagrangians.

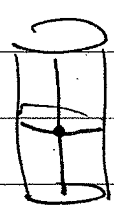
③ Q closed, smooth

$\cup$

N . $\rightsquigarrow$ conormal bundle of $N \subset T^*Q$

e.g. N = pt.

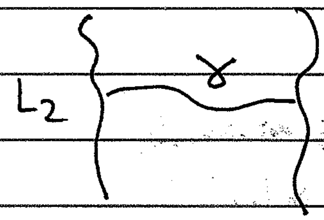
cotangent fibre

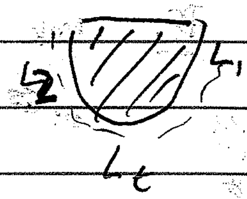


Q.

Given L_1, L_2, look at $\mathcal{P}(L_1, L_2)$ "$\longrightarrow$" $\mathbb{R}$

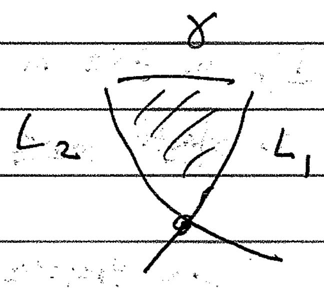
area of a half disc



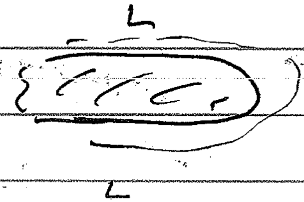


$\rightsquigarrow$ get $HF^*(L_1, L_2)$

Morse homology is Floer homology.

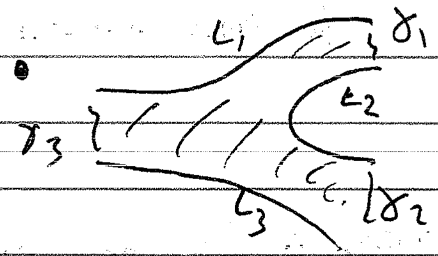


<u>Operations:</u>

- 

$\in HF^*(L, L)$

  e_L.

- 

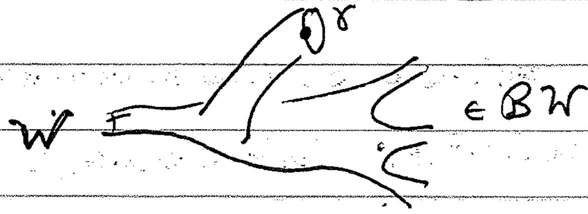
$\rightsquigarrow$ $HF^*(L_1, L_2) \otimes HF^*(L_2, L_3)$.

at chain-level, ~~not~~ associative $\rightsquigarrow$ get a full A_∞ structure.

There are maps

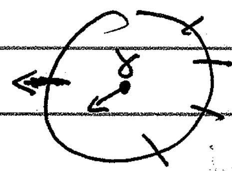
$$\begin{tikzcd} HH_*(W, W) \arrow[r] \arrow[rr, bend right] & SH^*\emptyset \arrow[r, "\mathcal{CO}"] & HH^*(W, W) \end{tikzcd}$$

$\mathcal{CO}$:



conformally

$\langle \sim \rangle$



Thm (A-Seidel) $\mathcal{CO}$ is an isomorphism if M is affine variety.

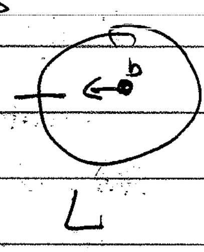
Thm: (Ganatra) The map $\mathcal{OC}$ is also an iso.

Recall: ~~there is always a map~~

$$HH^*(\mathcal{W}, \mathcal{W}) \downarrow HF^*(L, L).$$

Assume: L is compact, and $HF^*(L,L) \cong H^*(L)$.

Then, the composed map $SH^* \longrightarrow HF^*(L,L)$ is



~~Given $b \in SH$,~~

$b|_L \in H^*(L)$.

Now, say $b \in SH^1$ (corresponds to holo. vector fields on mirror).

Def: [Seidel-Solomon] L is b-invariant if $[b]|_L = 0$.
class in cohomology

Define an equivariant object to be a pair (L, c), where $\partial c = b|_L$.
at chain level.
(how action is trivialized).

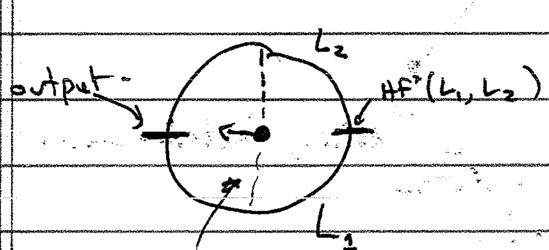
Given L_1, L_2 which are equivariant, we should have

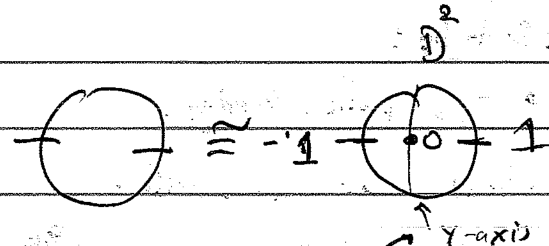
$\mathbb{K}$ alg. closed $\quad HF^*(L_1, L_2) = \bigoplus_{\lambda \in \mathbb{K}} HF^*_\lambda(L_1, L_2)$ ← weight decomposition.

(fixed pt. ⇝ action on tangent space ⇝ decompose tangent space by weight of actions).

eigenvalues of an operator $\boxed{b^1}$.

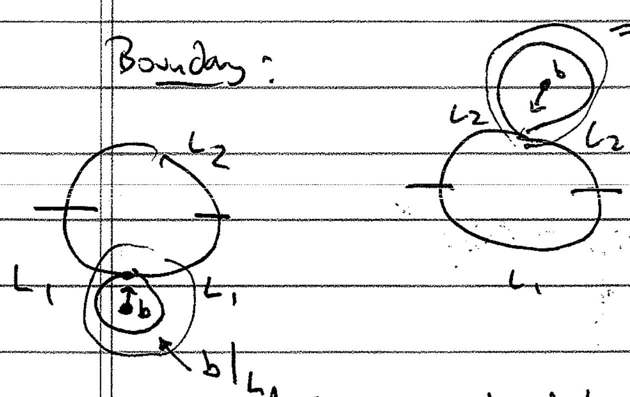
$b^1 : HF^*(L_1, L_2) \circlearrowleft$ degree 0. ambiguity: translation





1-parameter family of Riemann surfaces, varies along

Boundary:



this is a secondary operation.

Need to choose equivariant str. b/c need to know how you fill both boundaries.

Obs: [Seidel–Solomon] If b is mirror to a dilation of the holomorphic volume form, then the weight on $HF^n(L,L)$ is necessarily 1, ($HF^0(L,L)$ always weight 0).

Note that S^1 acts on $\mathcal{L}M$ by rotating loops

$\rightsquigarrow \Delta : SH^*(M) \longrightarrow SH^{*-1}(M)$

Claim: Mirror to a dilation is a class b satisfying

$$\Delta b = e \leftarrow \text{unit in } SH^0 \cong HH^*(\mathcal{W}, \mathcal{W}).$$

$$1$$

(Kontsevich: this is divergence in poly vector fields).

b vector field on X mirror variety

Dilation:

$$\mathcal{L}_b \Omega = \Omega.$$

Cartan formula:

$$\mathcal{L}_b \Omega = d(i_b \Omega) + \cancel{i_b(d\Omega)} = \Omega.$$

$$b \in H^*(X, \Lambda^* T) \xrightarrow{i_b \Omega} H^*(X, \Omega^*) \circlearrowleft \text{de Rham}$$

(contract w/ volume form)

Ω volume form

$\Rightarrow$ this is an iso.

Transport de Rham over here $\to$ get

$$H^*(X, \Lambda^* T) \circlearrowleft \Delta.$$

The thing that goes to Ω is e,

so we get $\Delta b = e$.

Not done, b/c we've not convinced ourselves that

Δ ~~is the~~ is the BV action in mirror symmetry.

$$H^*(X, \Omega^\bullet) \cong HH_*(\mathcal{A}_X, \mathcal{A}_X)$$

dg model for derived category of coherent sheaves

$$\mathcal{A}_X \otimes_{\mathcal{A}_X} \otimes_{\mathcal{A}_X} \mathcal{A}_X$$

diagonal bimodule

There's a $\mathbb{Z}/2$ action flip.

property: $\mathcal{A}_X \otimes_X \mathcal{A}_X = \mathcal{A}_X$.

So, we could split

$$\mathcal{A}_X \underbrace{\mathcal{A}_X \otimes \mathcal{A}_X \otimes \cdots \otimes \mathcal{A}_X}_{k \text{ copies}} \cong \mathcal{A}_X \otimes \mathcal{A}_X$$

$\mathbb{Z}/k\mathbb{Z}$ action

These actions are compatible as $k \to \infty$
$\Rightarrow S^1$ action, induces an operator on HH_* $\circlearrowleft \Delta$. related to cyclic homology.
identical to de Rham operator.

Symplectically:
Let $L_M \subset M^+ \times M^-$ be the diagonal.

Given $L_1 \subset M \times M$, $L_2 \subset M \times M$ $\Big\} \to L_1 \circ L_2 \subset M \times M$.

Fibre product over diagonal in the middle two, circled.

$L_1 \times L_2 \subset M \times \underbrace{M \times M} \times M$

$\Delta \circlearrowright SH^*(M)$

Point: $SH^*(M) \cong HF^*(L_M, L_M)$.

$\mathbb{Z}/4\mathbb{Z} \circlearrowright \cong HF^*(L_M \circ L_M, L_M \circ L_M)$

$\mathbb{Z}/2k\mathbb{Z} \circlearrowright \cong HF^*((L_M \circ \cdots \circ L_M)^k, (L_M \circ \cdots \circ L_M)^k)$.

In the inverse limit over k, obtain

circle action, operator on SH^* agrees with Δ.

Outcome: Compatibility of d on X,
Δ on M,

follows from:

(1) Mirror symmetry for X and M.

(2) Fourier–Mukai calculus on the two sides.

symplectic formalism is due to Wehrheim–Woodward, Ma'u.

This theory has applications:

(1) Lagrangian embeddings (Seidel–Solomon, Seidel)

(2) (A–Smith): Khovanov homology, from a symplectic point of view. (Prove a formality of Fukaya categories).

Kontsevich "Formality from purity"