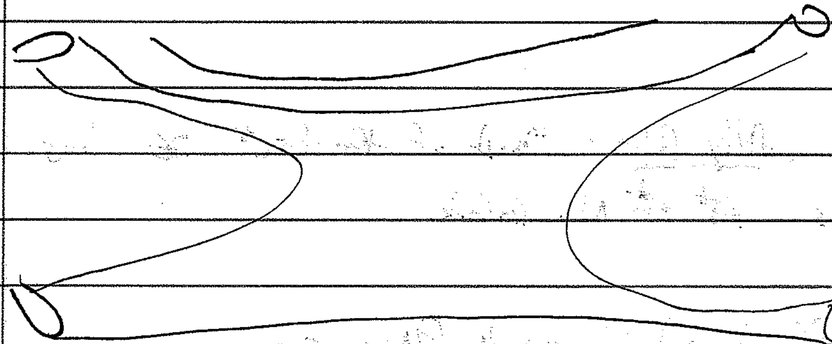
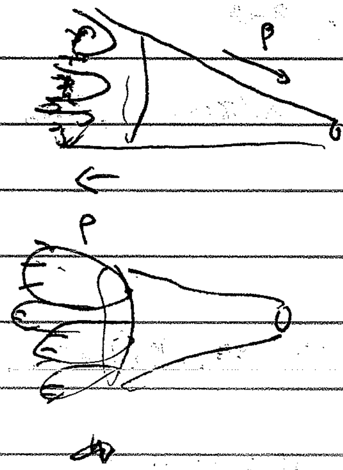




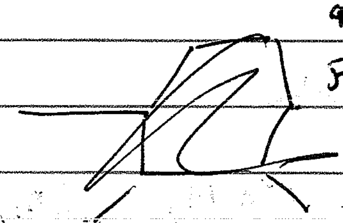
$F^*: W^{in}\text{-mod} \to W\text{-mod}.$

$F: W \to W^{in}$

and

$F_*: W\text{-mod} \to W^{in}\text{-mod}$

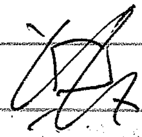
$\otimes F.$

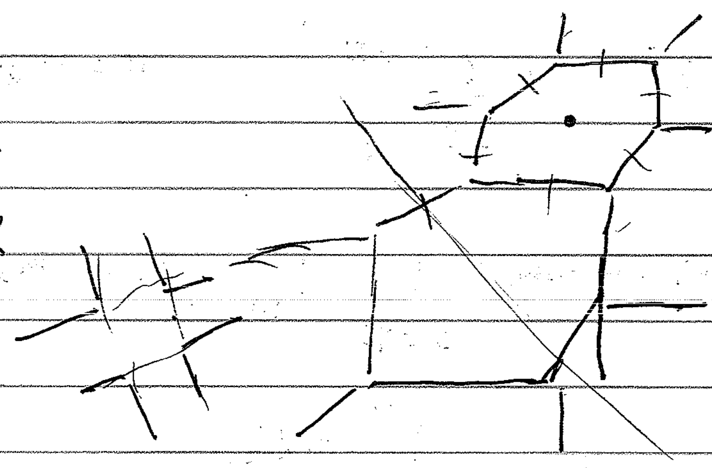


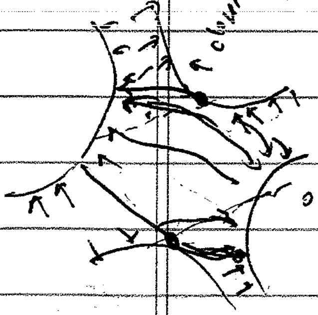
$j^*: X \to U$

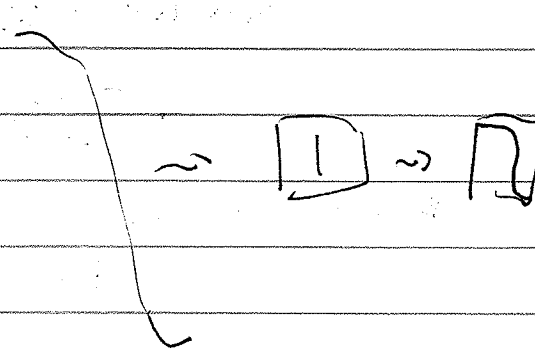
$j_!: U \to X$

?









$F^*(T^*X) \otimes \to F^*(M)$

$F(D^*(\mathbb{R}^n)) \to F^*(T^*(Y)) \to$

$:= F(M).$

90° Constr(X)

Constr$(\mathbb{R}^n) \to$ Constr(Y)

Abouzaid III:

$\mathcal{A}$ A_∞ ~~element~~ alg. $\{m_d \mid d \geq 1, m_1^2 = 0\}$

$H^\cdot(\mathcal{A})$ graded associative (i.e. also A_∞, but $m_d = 0$ unless $d = 2$)

Def: $\mathcal{A}$ is formal if $\exists$ an A_∞ equiv. $\mathcal{A} \xrightarrow{\sim} H^*(\mathcal{A})$

criterion for formality:

A graded associative algebra:

Obs: $b_\chi : A \to A$ Euler field

$$b_\chi(x) = \deg(x) \cdot x$$

Because $\deg(x \cdot y) = \deg(x) + \deg(y)$

show that b_χ defines a class in $HH^1(A, A)$

$$b_\chi(x \cdot x) - x \cdot b_\chi(y) - b_\chi(x) \cdot y) = 0$$

Assume we have $b \in CC^1(\mathcal{A}, \mathcal{A})$ closed

$$= \mathrm{Hom}\left(\bigoplus_{i=0}^{\infty} \mathcal{A}[1]^{\otimes i}, \mathcal{A}\right)$$

$(b^0, b^1, b^2, b^3, \sim)$ terms

s.t. (1) $b^0 = 0 \in \mathcal{A}$ (can be weakened)

(2) b^1 induces the euler vector field on $H^\cdot(\mathcal{A})$.

Then, $\mathcal{A}$ is formal (over char. 0 field)

(In fact, $\exists$ bijection between classes b satisfying this property and q.i. from $\mathcal{A} \to H^\cdot(\mathcal{A})$ up to equivalence)

It's a formality iff $\exists$ such ~~a b~~.

{b = 0 case is ok, b/c A_∞ alg. supported in deg-0 is formal}

Idea of pf: Consider an action of k^* on $\mathcal{A}$, by t^{i} on $\mathcal{A}^i$ ← ith graded piece

Conjugate operation m_d ← A_∞ operation by this action.

<u>observation</u>: If $m_d = 0$ unless $d = 2$, then A_∞ str. is fixed by this action.

"existence of b tells you you're a fixed pt. of this action, up to higher order terms."

Use higher order terms of b to find formal diffeomorphism, and obtain A_∞ str. which is k^*-invariant.

So, "Purity ⟹ Formality."

↑ class in HH^1 induces an action on $H^\cdot(\mathcal{A})$. Purity is the condition that this action is the Euler v. field.

Examples: ① DGMS

Deligne–Griffiths–Morgan–Sullivan:

If Q cpct. Kähler, then $\mathcal{F}^\cdot(Q, Q)$ is formal.

We know $\mathcal{E}^\cdot(Q, Q) \underset{A_\infty \text{ equiv.}}{\simeq} \mathcal{F}^\cdot(Q, Q)$, $Q \subset T^*Q$.

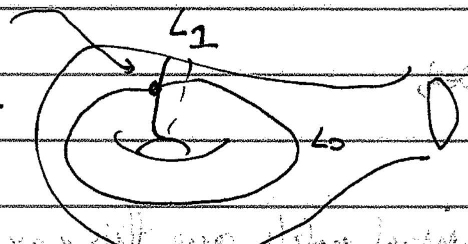
(Harrison cohomology & HH' ← naturally mapping one into other –

② Note that $\boxed{S^1}$ is also formal. (compute by hand)

Lekili–Perutz:

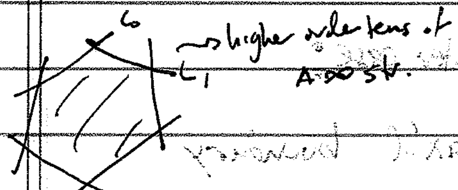
$$T^2 \setminus \{pt.\} = M.$$

all discs are constant



$$\mathcal{A} \subset \mathcal{F}(M)$$

$\uparrow$ subcategory w/ objects L_0, L_1.



$H^*(\mathcal{A})$ is a quiver w/ relations

$$\begin{tikzcd} L_0 \arrow[r, bend left, "u"] & L_1 \arrow[l, bend left, "v"] \end{tikzcd}$$

$$uvu = 0$$
$$vuv = 0.$$

<u>Thm</u> (L–P) $\mathcal{A}$ is <u>not</u> formal

(over char. 0 field, cannot eliminate m_6 and m_8).

Can take other punctures, & same thing holds.

Non-trivial example of formal algebra appearing in Fukaya categories.

(Motivation: Khovanov homology)

(joint w/ Ivan Smith)

some open locus, milnor fibre.

$$\mathcal{Y}_n = \mathrm{Hilb}^0_n(A_{2n-1})$$

$$\mathbb{C}^3 \supset A_{2n-1} = \{p(z) = x^2 + y^2\}$$

$\deg(p) = 2n$. w/ simple roots.

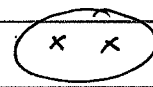
$$\begin{tikzcd} A_{2n-1} \arrow[d] \\ \mathbb{C} \end{tikzcd}$$

Hilb^0: Consider only those subschemes whose projection to $\mathbb{C}$ also has length n.

(an affine open subset of $\underline{\mathrm{Hilb}}$).

e.g. $\mathcal{Y}_1 = \mathrm{Hilb}_1^{\circ}(A_1) = A_1 = T^*S^2$.

$$\begin{array}{c} A_1 \\ \downarrow z \\ \mathbb{C} \end{array}$$

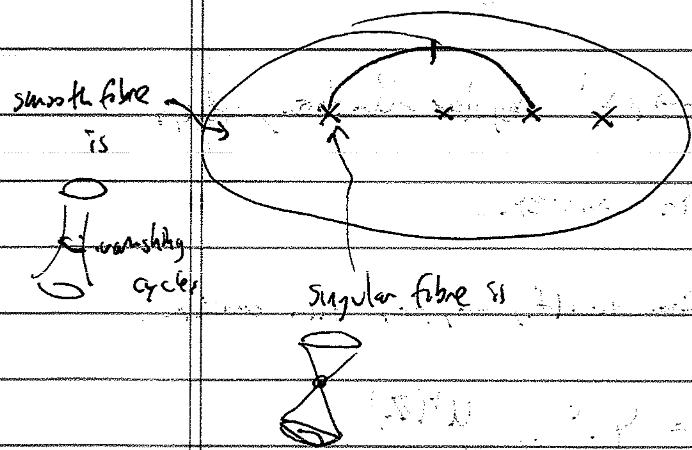


Goal: produce an interesting collection of Lagrangians in $\mathcal{Y}_n$.

– First step: produce Lagrangians in A_{2n-1}.

(Donaldson, Seidel : matching spheres).

Assume that all roots of p are real. (symplectically, this is ok)



For each embedded path γ connecting roots of p, obtain a Lagrangian sphere $L_\gamma \subset A_{2n-1}$.

Union of all vanishing cycles over the arc.

$$\begin{array}{ccc} \mathrm{Hilb}_n^{\circ}(A_{2n-1}) & & (A_{2n-1})^n \\ & \searrow^{\text{Hilbert–Chow}} \quad \swarrow & \\ & \mathrm{Sym}^n(A_{2n-1}) & \end{array}$$

Away from diagonal, this is bijective.

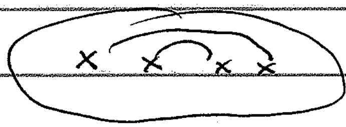
So, can construct Lagrangians away from diagonal.

$\mathfrak{C}$:= crossingless matching

Given n non-intersecting paths in upper half plane connecting the zeroes of p, obtain a Lagrangian $L_{\mathfrak{C}} \subset \mathcal{Y}_n$.

$\cong$ topologically $(S^2)^n$, so $HF^*(L_{\mathfrak{C}}, L_{\mathfrak{C}}) \cong H^*(S^2)^{\otimes n}$.

In our case, have



and

$\mathcal{A}$ is the category of Lagrangians in $\mathcal{Y}_n$ which are obtained from matchings.

(only finitely many objects ~ Catalan #).

(all of these are exact Lagrangians in affine).

Observation: (Seidel-Smith) At the level of cohomology, "this looks like" Khovanov's arc algebra.

(Btw, $\mathcal{Y}_n$ is a nilpotent slice ⇝ related to representation theory of sl_2 "2×2 blocks in sl_{2n}")

Khovanov: The arc algebra has "many" non-trivial deformations which preserve multiplication but introduce higher order multiplications. (computation of HH^2, rk goes to ∞)

~~Later~~ (in fact, ∃ sympl. manifolds w/ this algebra, non-formal).

A-Smith: $\mathcal{A}$ is formal over $\mathbb{Q}$

Main idea: Construct class b in $SH^1(\mathcal{Y}_n)$, and show that the formality criterion holds.

b comes from a "properification" of $\mathcal{Y}_n \longrightarrow \mathrm{Conf}_n(\mathbb{C})$. coming from (properification of)

$A_{2n-1} \longrightarrow \mathbb{C}$.