

Example:

Thm: $(B, \mathcal{V}B, k)$ X quasi-compact quasi-spc k .

$$D_{qc}(\mathcal{O}_X - Mod) \cong D(A)$$

$$E \in \text{Perf}(X)$$

$$\text{Perf}(X) \cong \text{Perf}(A)$$

$$A = \text{R}\Gamma(E)$$

$$F \mapsto \text{R}\Gamma(E, F)$$

X finite type k -field

sp.

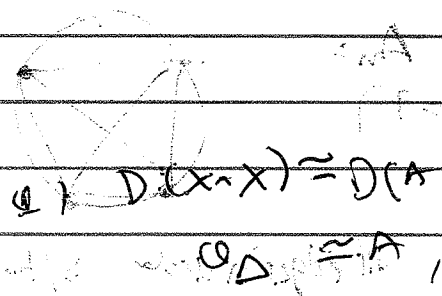
fix.

Prop: $\text{Perf}(X) = \text{Perf}(A)$

(1) X smooth $\Leftrightarrow A$ smooth

(2) X prop $\Leftrightarrow A$ prop

easy exercise.



Def: (1) A smooth if $A \in \text{Perf}(A \otimes A^{op})$

(2) A prop if $A \in \text{Perf}(k)$.

$D^b(\text{coh})$ now.

Thm: X separated, finite type k , k -perfect field. [Lunts]

$$D^b(\text{coh}(X)) \cong \text{Perf}(A),$$

and A -smooth

Pf: Identity $\text{Perf}(A \otimes A^{op}) \cong D^b(X \times X)$

but need to check

$$A \cong \Delta_* \mathbb{P}$$

some dualizing complex \mathbb{P}

D identity $A^{op} \in X$

$$D \in D^b(\text{coh}(X))$$

dualizing complex with diagonal embeddings

(k perfect $\Rightarrow X$ has a smooth strat, otherwise only regular by finite type)

\Rightarrow need to go to $X \times X$.

homotopy
equivalence

subalgebra of the algebra, same as $D(A)$

Recall, $D^0(X) \cong D(A)$

↑
homotopy cat

$K(\text{Inj}_X)$

↑
injection quasi-isomorphisms

if X not smooth, different from $D^b(X)$

compactly gen. by $D^b(\text{coh}(X))$

Summary: $\text{Per}(X)$ reflects properties of X , but $D^b(\text{coh})$ doesn't.

$D^b(\text{coh})$ always smooth, but when X singular, unbounded to the right.

homotopy finiteness (hfp) homotopically fin. presites

Def: A dga / k

(1) A is hfp if A is homotopically compact

i.e. $\text{hocolim } \text{Hom}(A, B_i) \xrightarrow{\sim} \text{Hom}(A, \text{hocolim } B_i)$

simplicially enriched category, or hom in hfp category of dgas
reasonable in any cofibrantly compactly gen. model cat.

(2) B is finite cellular if

$B \in \langle X_0, \dots, X_k \rangle$, $dX_i \in \langle X_0, \dots, X_{i-1} \rangle$

A is hfp if have

$A \rightarrow B \rightarrow A$ for some cellular B

in $\text{Ho}(\text{dg-alg}_k)$

Thm: (Toën) 1) hfp \Rightarrow smoothness

(\mathcal{H}_c for cellular, easy to resolve Δ ,

and smoothness preserved under projectives)

2) smooth + proper \Rightarrow hfp.

3) $D(A) \simeq D(A')$ Morita equiv,
 (equivalent to a map between Proj , internal modules)

$A \text{ hfp} \Leftrightarrow A' \text{ hfp}$.

Also, do this in:

$\text{Ho}(\text{dg-cat } k)$, where each equiv \Rightarrow Morita equiv.

(where each cell $\text{dg-cat} = \text{proj } k$ algebra of k -modules)

Since they are generated by one object

quotient by one object. $\in \text{Ob}(C)$.

Thm (E, Toën?) 1) If C dg-cat which hfp $\Rightarrow C/E$ is hfp.

2) $C = \langle A, B \rangle$ semi-orthogonal

splitting $M_{AB}(X, Y) = \text{Hom}(X, Y)$ all maps from A to B .

GT pretriangulated A, B -hfp, $M \in \text{Perf}(A \oplus B^{\text{op}}) \Rightarrow C$ -hfp.

Def: A functor $F: C \rightarrow T$ is a smooth compactification

if C is smooth and proper, and F is a localization (up to direct summands) (every object of T is a summand of some thing in image)

and $\text{Ker}(F)$ gen. by one object as a triangulated category.

Cor: If T has a smooth compactification, then T is hfp.

(expect: any hfp object has a smooth compactification)

Think: If X sp. finite type / $\text{char}(k) = 0$, then

$\rightarrow D^b(\text{coh}(X))$ has a smooth compactification, i.e. a dg functor
 $C \rightarrow D^b(\text{coh}(X))$

$\langle D^b(Y_1), \dots, D^b(Y_n) \rangle$

all Y_i are smooth & proper.

(2) $W \in \mathcal{O}(X)$, then \rightarrow for $\text{MF}_{\text{coh}}(X, W)$

Here, $C = \langle \text{MF}_{\text{coh}}(Y_i, W) \rangle$

Y_i are smooth, and $\text{supp}(W) = Y \rightarrow \mathbb{A}^1$ proper (enough to require crit proper),
 (Bridgeland)

Motivation: Case : If $X \rightarrow Y$ res. of singularities, and

$Rf_* \mathcal{O}_X \simeq \mathcal{O}_Y$ (so Y has rational singularities), then

$Rf_* : D^b(X) \rightarrow D^b(Y)$ is a localization.

In general, this remains open - (very hard to control kernel of such a functor).

Expected for MF as well: $\text{MF}_{\text{coh}}(X, f^*W) \rightarrow \text{MF}_{\text{coh}}(X, W)$

Uses categorical res. of singularities (Kawamata, Matsuda).

Case : $Y = \text{Spec } A$, $I \subset A$, $I^n = 0$

Construct cat.

$$\text{Ob}(C_A) = \{t, \dots, n\}, \text{ with } \text{hom}(i, j) = \begin{pmatrix} A/I & I/I^2 & \dots & I^{n-1} \\ 1 & A/I^2 & & I \\ & & \ddots & \\ A/I & A/I^2 & & A \end{pmatrix}$$

$I^{\max(j-i, 0)}$

I

(understand - $\text{Hom}(A, A)$ algebra)

Consider A as a cat. of one object, $\text{Hom}(A, A) = \text{Hom}(A, A)$

$$\pi: A \rightarrow CA \quad \text{with } \pi(x) = (x, x)$$

Get $\pi^*: \text{Perf}(A) \rightarrow \text{Perf}(CA)$ very faithful

and $\pi_*: D^b(CA) \rightarrow D^b(A)$

if A is finite type (k , this is a localization) (check explicitly)
kernel generated by modules supported on other vertices

like quiver \dots It's a mixed space, b/c have

$$\text{And, } D^b(CA) = \langle D^b(A/I), \dots, D^b(A/I) \rangle$$

i.e. if $A/I \sqsupset \text{smooth}$, get decomposition into $D^b(\text{smooth})$

S separated of finite type (k)

sheaf of ideals $I \subset \mathcal{O}_S$, $I^2 = 0$

then construct a new rigid space $\tilde{S} \rightarrow S$ by sheafifying this structure
smooth if quotient is smooth

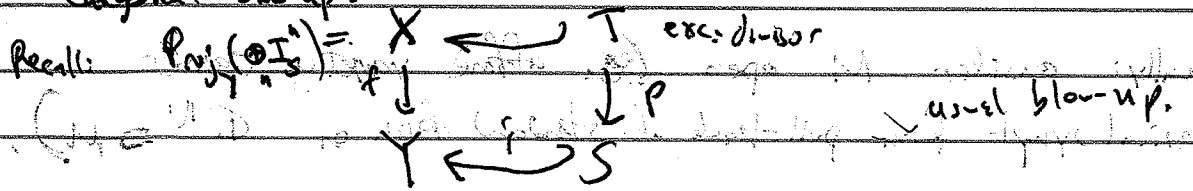
$$D^b(\text{coh}(\tilde{S})) = \langle D^b(\text{coh}(S_0)), \dots, D^b(\text{coh}(S_n)) \rangle$$

localization is easy, b/c functor is controlled by Γ -structure
check abelian category, actual kernel

Problem: singularities of the reduced part.

(Categorical) blow-up

Categorical blow-up:



Assume $R_f(I^n) \cong I_S^n$ (Serve: holds for $n \gg 0$ b/c $I^n = \mathcal{O}_{X/Y}(n)$ for $n \gg 0$)

Assumption: $\forall n \gg 1$ (satisfiable, b/c we replace f by inf. above, in which blow-up is same??)

~~Categorical blow-up~~ Define: categorical blow-up

$$\mathcal{D} = \langle D^b(S), D^b(X) \rangle$$

$$= D^b(S) \times D^b(X)$$

$\xrightarrow{p_{X^*}}$
glued along this factor, via

mutation $= D^b(X) \times_{j_* p^*} D^b(S)$

\uparrow
right adjoint gluing
bimodule

Recall: if have $A, B, M \in \text{Per}(A \otimes B^{\text{op}})$

can define gluing $A \times_B B$ dg cat
(w/ semi-ortho decomp)

$\langle A, B \rangle, M \in \mathcal{B}$
have from A to B
(define M (glue with for explicit abs., the following table with).

identifies $M \cong M \otimes_B M$

Then, we can write a functor

$\pi_* : \mathcal{D} \rightarrow \mathcal{D}^b(X)$ $D(B) \rightarrow D(A)$

as a triple $(F_X, F_S, \eta = F_X \rightarrow j_* p^* F_S)$ Fiber := Core $[-1]$ of a map from

$\pi_* (F_X, F_S, \eta) = \text{Fiber}(f_* F_X \xrightarrow{w} i_* F_S)$

where $f_* F_X \xrightarrow{f_* \eta} f_* j_* p^* F_S = i_* p_* p^* F_S \rightarrow i_* F_S$

(a) Thm 4.1 $\mathcal{D} \rightarrow \mathcal{D}^b(Y)$ is localization

above, there is a natural functor $\phi: \mathcal{D}(T) \rightarrow \mathcal{D}$

$$\phi(G) = (\mathcal{I}_X G, \mathcal{P}_X G, \mu_G)$$

$$\mu_G: \mathcal{I}_X G \rightarrow \mathcal{I}_X \mathcal{P}_X G$$

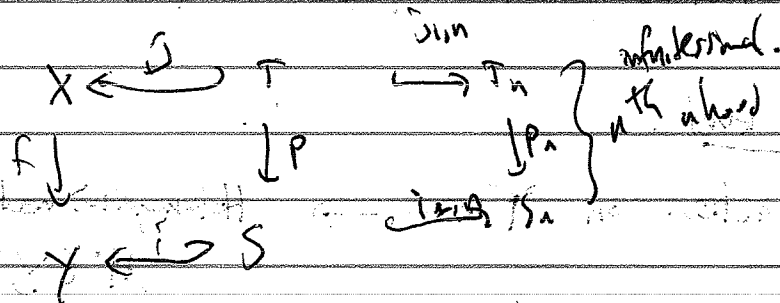
Easy to show

$$\pi_X \phi = 0$$

(b) If $G \in \mathcal{D}(T)$ - generator, then $\phi(G)$ - generator of $\ker \pi_X$

How?

Ex: See picture, but



$$Rf_* (\mathcal{I}_T^n) \cong \mathcal{I}_S^n, n \geq 0$$

if

Thm: $\mathcal{P}_X: \mathcal{D}^b(T) \rightarrow \mathcal{D}^b(S)$ is loc $\Leftrightarrow f_*: \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(Y)$ - loc, as

$$\ker f_* = \langle \mathcal{I}_X \ker \mathcal{P}_X \rangle$$

Prop: If $f_*: \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(Y)$ - loc.

\Downarrow

$f_*: \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(Y)$ - loc, w/ given level (open on T)

Prop: Assume $\mathcal{P}_n: \mathcal{D}^b(T_n) \rightarrow \mathcal{D}^b(S_n)$ - loc, w/ $\ker \mathcal{P}_n = \mathcal{I}_{T_n} \times \ker \mathcal{P}_X$

Then, $f_*: \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(Y)$ - loc, $\ker f_* = \langle \mathcal{I}_X \ker \mathcal{P}_X \rangle$.

why? $D_T^b(X) = \text{colim } D^b(T_n)$

$D_S^b(Y) = \text{colim } D^b(S_n), \mathbb{G}$

loc. annulus of colims.

Now, reduced to problem of showing each part is localizable & central kernel!

By induction:

Inductive step:

Suppose $D^b(T_n)$

$\downarrow P_n \#$
 $D^b(S_n)$

localization

\Rightarrow

$D^b(T_{n+1}) \leftarrow D^b(T_n)$

part $\#$ also localizable $\downarrow P_{n+1} \#$

$D^b(S_{n+1}) \leftarrow D^b(S_n)$
 π_S

Inductive inductive process

is compatible w/ S.O.D.

\Rightarrow it also

part of
rather