Efimov:

Thm: (B, VdB, K) X-quasi-comp. quasi-sep sch /k.

$D_{qc}(\mathcal{O}_X\text{-Mod}) \cong D(A)$ $\qquad E \in Perf(X)$

$Perf(X) \cong Perf(A)$ $\qquad A = REnd(E)$

$F \longmapsto RHom(E, F)$

X finite type /k-field

sep.

Fix.

Prop: $Perf(X) = Perf(A)$

(1) X-smooth $\Leftrightarrow$ A-smooth $\qquad$ (1) $D(X \times X) \cong D(A \otimes A^{op})$

$\mathcal{O}_\Delta \cong A$, preserves smoothness

(2) X-prop $\Leftrightarrow$ A-prop

easy exercise.

Def (1) A smooth if $A \in Perf(A \otimes A^{op})$

(2) A prop if $A \in Perf(K)$.

$D^b Coh$ now.

Thm: X separated, finite type /k, k-perfect field. [Lunts]

$D^b Coh(X) \cong Perf(A)$,

and A-smooth

Pf: Identify $Perf(A \otimes A^{op}) \cong D^b(X \times X)$ but need to choose $\mathcal{D}$

$A \cong \Delta_* \mathcal{D}$ $\qquad$ some dualizing complex on X

$\boxed{\mathcal{D} \text{ identifies } A^{op} \text{ w/ } X}$

$\mathcal{D} \in D^b Coh(X)$

dualizing complex w.r.t. diagonal embedding

(k perfect $\Rightarrow$ X has a smooth point, otherwise only regular by finite type)

used to go to $X \times X$.

Recall, $D^{co}(X) \simeq D(A)$

$\uparrow$

homotopy cat

$K(Inj_X$

$\uparrow$ injective quasi-coh on X)

if X not smooth, different from $D^b(X)$

compactly gen by $D^b Coh(X)$.

Summarize: $Perf(X)$ reflects properties of X, but $D^b Coh$ doesn't.

$D^b Coh$ always smooth, but when X singular, but unbounded to the right

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homotopy finiteness (hfp) homotopically fin. presented

Def: A dga $/k$.

(1) A is hfp if A is homotopically compact

i.e. $\operatorname{hocolim} \operatorname{Hom}(A, B_i) \xrightarrow{\sim} \operatorname{Hom}(A, \operatorname{hocolim} B_i)$

simplicially enriched category, or hom in htpy category of dgas

(reasonable in any cofibrantly compactly gen. model cat.)

(2) B is finite cellular if

$B = k\langle x_1, \dots, x_k\rangle$, $dx_i \in k\langle x_1, \dots, x_{i-1}\rangle$

A is hfp iff have

$A \to B \to A$ for some cellular B

id

in $Ho(dg\text{-}alg_k)$

Thm: (Toën) 1) hfp $\Rightarrow$ smoothness.

(b/c trivial for cellular, easy to resolve Δ,

and smoothness preserved under projections)

2) smooth + proper $\Rightarrow$ hfp.

3) If $D(A) \simeq D(A')$ Morita equiv, (equivalent to a map between Perfs, small modules)

A hfp $\Leftrightarrow$ A' hfp.

Also, do this in:

$Ho(dg\text{-}cat_k)$, where weak equiv = {Morita eq.}.

(where finite cell dg cat = path algebra of a quiver)

Since they're b/c generated by one object

quotient by one object. $E \in Ob(C)$.

Thm (E, Toën?) 1) If C dg cat which hfp $\Rightarrow$ C/E is hfp

2) $C = \langle A, B \rangle$ semi-orthogonal

gluing kernel $M_{AB}(X, Y) = hom(X, Y)$ all homs from A to B.

(if pretriangulated dg) A, B - hfp, $M \in Perf(A \otimes B^{op})$ $\Rightarrow$ C - hfp

Def: A functor $F: C \to T$ is a smooth compactification if C is smooth and proper, and F is a localization (up to direct summands) (every object of T is a summand of something in image), and $Ker(F)$ gen. by one object as a triangulated category

Cor: If T has a smooth compactification, then T is hfp.

(expect: any hfp dgcat has a smooth compactification)

Thm: (1) If X sep. finite type / ~~k~~, char$(k)=0$, then $D^b Coh(X)$ has a smooth compactification, i.e. a dg functor

dgcat $\to$

$$C \longrightarrow D^b Coh(X)$$

$$\langle D^b(Y_1), \dots, D^b(Y_m)\rangle$$

all Y_i are smooth & proper.

(2) $W \in \mathcal{O}(X)$, then — " — for $MF_{Coh}(X, W)$

Here, $C = \langle MF_{Coh}(Y_i, w)\rangle$

Y_i are smooth, and $w_i: Y \to \mathbb{A}^1$ proper (enough to require crit proper).

[Bondal-Orlov]

Motivating Conj: If $X \to Y$ res. of singularities, and

$Rf_* \mathcal{O}_X \simeq \mathcal{O}_Y$ (so Y has rational singularities), then

$Rf_*: D^b(X) \to D^b(Y)$ is a localization.

In general, this remains open. (very hard to control kernel of such a functor).

Expected for MF as well: $MF_{Coh}(X, f^* W) \to MF_{Coh}(X, w)$.

Uses categorical res. of singularities (Kuznetsov, Lunts).

Case: $Y = \operatorname{Spec} A$, $I \subset A$, $I^n = 0$.

Construct cat.

$$Ob(C_A) = \{1, \dots, n\}, \text{ with } \hom(i,j) = \begin{pmatrix} A/I & I/I^2 & \cdots & I^{n-1} \\ \vdots & A/I^2 & & I \\ \vdots & \vdots & & I \\ A/I & A/I^2 & & A \end{pmatrix}$$

$$= \frac{I^{\max(j-i,0)}}{I^{j}}$$

Consider A as a cat. w/ one object,

$$\pi : A \longrightarrow C_A$$
$$\bullet \longmapsto n$$

Get: $\pi^* : \mathrm{Perf}(A) \longrightarrow \mathrm{Perf}(C_A)$ fully faithful.

and $\pi_* : D^b_{f.g.}(C_A) \longrightarrow D^b_{f.g.}(A)$ ← fin. gen.

If A is finite type/k, this is a localization (check explicit)

kernel generated by modules supported on other vertices

like quiver ... It's a ringed space, b/c have quiver

And, $D^b(C_A) = \langle D^b(A/I), \ldots, D^b(A/I) \rangle$

i.e. if A/I is smooth, get decomposition into D^b(smooth, ...).

S separated of finite type /k.

sheaf of ideals $I \subset \mathcal{O}_S$, $I^n = 0$.

then construct a new ringed space $\tilde{S} \to S$ by sheafifying this construction

smooth if quotient is smooth

$$D^b \mathrm{Coh}(\tilde{S}) = \langle D^b_{coh}(S_0), \ldots, D^b \mathrm{Coh}(S_0) \rangle$$

localization is easy, b/c functor is controlled by t-structure

check abelian category, actual kernel.

Problem: singularities of the reduced part.

Categorical blow-up:

Recall: $\mathrm{Proj}_Y(\bigoplus_n I_S^n) = X \hookleftarrow T$ exc. divisor

$$\begin{tikzcd} X \arrow[d, "f"] & T \arrow[l, hook'] \arrow[d, "p"] \\ Y & S \arrow[l, hook', "i"] \end{tikzcd}$$

usual blow-up.

Assume $Rf_*(I_T^n) \simeq I_S^n$ (Serre: holds for $n>>0$, b/c $I_T^n = \mathcal{O}_{X/Y}(n)$ for $n>>0$).

Assumption: $\forall\, n \geq 1$.

(satisfiable, b/c can replace S by inf. nbhd, on which blow-up is same ??)

~~Categorical blow-up~~ Define: categorical blow-up

$$\mathcal{D} = \langle D^b(S), D^b(X) \rangle$$

$$= D^b(S) \times D^b(X)$$

$\xrightarrow{p_* j^*}$ glued along this functor, via

mutation $= D^b(X) \times_{j_* p^!} D^b(S)$

right adjoint gluing bimodule.

Recall: If have A, B, $M \in P(A \otimes B^{op})$, can define gluing $A \times_M B$ dg cat w/ semi-orth. decomp. $\langle A, B \rangle$, M is homs from A to B. ~~(details here~~ (skew - with for explicit algs, take triangular-tab with).

identify $M \simeq M \otimes_B^L -$: $D(B) \to D(A)$

Then, we can write a functor

$$\pi_* : \mathcal{D} \longrightarrow \mathcal{D}^b(Y)$$

as a triple $(F_X, F_S, u : F_X \to j_* p^! F_S)$. Fiber := Cone $[-1]$ of a morphism

$$\pi_*(F_X, F_S, u) = \mathrm{Fiber}(f_* F_X \xrightarrow{\hat{u}} i_* F_S)$$

where $\underbrace{f_* F_X \xrightarrow{f_*(u)} f_* j_* p^! F_S = i_* p_* p^! F_S \to i_* F_S}_{\hat{u}}$

Thm: (1) $\pi_*: \mathcal{D} \to D^b(Y)$ is a localization

Have a natural functor $\phi: D^b(T) \to \mathcal{D}$

$$\phi(G) = (j_* G,\ p_* G,\ u_G),$$

$$u_G: j_* G \to j_* p^! p_* G.$$

Easy to show

$$\pi_* \phi = 0.$$

(2) If $G \in D^b(T)$ - generator, then $\phi(G)$ - generator of $\ker \pi_*$.

How?

Ex. Same picture, but

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, hook', "j"'] \arrow[d, "p"] \arrow[r, hook, "j_{1,n}"] & T_n \arrow[d, "p_n"] \\ Y & S \arrow[l, hook', "i"'] \arrow[r, hook, "i_{1,n}"] & S_n \end{tikzcd} \quad \Big\} \text{ infinitesimal } n^{th} \text{ nbhd}$$

Assume $\quad Rf_*(I_T^n) \cong I_S^n,\ n \geq 0$

Thm: if $p_*: D^b(T) \to D^b(S)$ is a loc $\Rightarrow f_*: D^b(X) \to D^b(Y)$ - loc, and

$\ker f_* = \langle j_* \ker p_* \rangle$.

Prop: If $f_*: D^b_T(X) \to D^b_S(Y)$ - loc.

$\Downarrow$

$f_*: D^b(X) \to D^b(Y)$ - loc, w/ same kernel (supp on T),

Prop: Assume $p_{n*}: D^b(T_n) \to D^b(S_n)$ - loc, and $\ker p_{n*} = j_{1,n*} \ker p_*$

Then, $f_*: D^b_T(X) \to D^b_S(Y)$ - loc, $\ker f_* = \langle j_* \ker p_* \rangle$.

why? $D^b_T(X) = \mathrm{colim}_n D^b(T_n)$

$D^b_S(Y) = \mathrm{colim}_n D^b(S_n)$, &

loc. commutes w/ colims.

Now, reduced to problem of showing each p_n is localization & control kernel:

By induction:

Inductive step:

partial rostrs

Suppose
$$\begin{array}{ccc} D^b(T_n) & & \\ \downarrow p_{n*} & \text{localization} & \\ D^b(S_n) & & \end{array} \Rightarrow \begin{tikzcd} D^b(T_{n+1}) \arrow[d, "p_{n+1*}"] & D^b(\widetilde{T_{n+1}}) \arrow[l, "\pi_T"'] \arrow[d, "\widetilde{p}_{n+1*}"] \\ D^b(S_{n+1}) & D^b(\widetilde{S_{n+1}}) \arrow[l, "\pi_S"] \end{tikzcd}$$

p_{n+1*} also loc.

~~inductive~~ inductive pieces

π also compatible w/ S.O.D.

$\Rightarrow$ π also loc.